Group Foliation and Moving Frames

Workshop on Moving Frames in Geometry
Centre de Recherches Mathématiques
Montréal, Canada

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Outline

Classical method of group foliation

Equivariant moving frames

Group foliation via moving frames

Introduction and motivation

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Non-invariant solutions: orbits of larger dimension.

Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung, **Lie 1895**

Sur l'intégration des systèmes différentiels qui admettent des groupes continus de transformations, **Vessiot 1904**

⋮

Group analysis of differential equations, **Ovsiannikov 1978**

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Given: a PDE and some symmetry group/pseudo-group.

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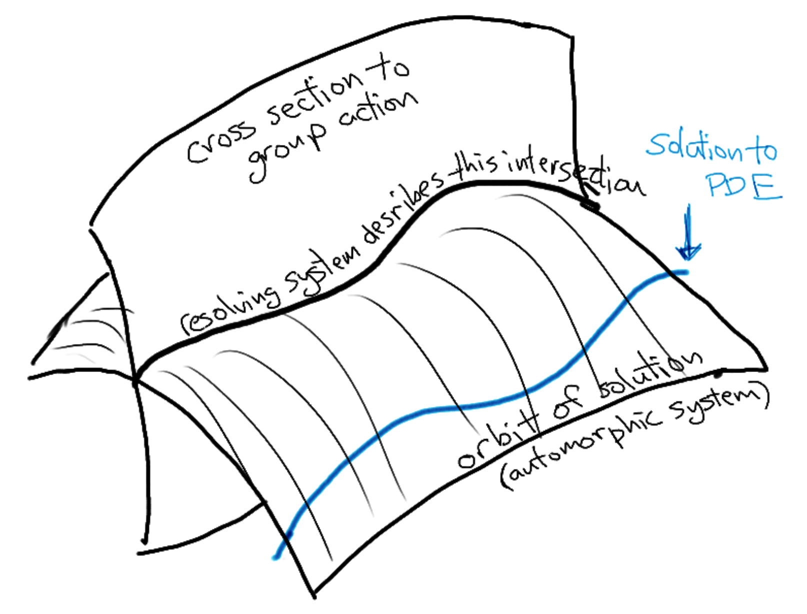
Group foliation: split the original equation into

automorphic system and **resolving system**.

Geometric idea



Example (a nonlinear heat equation)

The equation

$$u_t = u_{xx} - \frac{u_x^2}{u} \qquad \text{(NLH)}$$

admits the scaling symmetry

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A generating set of invariants is

$$x \qquad t \qquad I = \frac{u_x}{u} \qquad J = \frac{u_t}{u}.$$

An **automorphic system** is the orbit of a generic solution in $\mathcal{J}^1$:

$$I = \phi(x,t) \qquad J = \psi(x,t).$$

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If $\phi(x,t)$ is a solution to the heat equation above, then solving

$$\frac{u_x}{u} = \phi(x,t) \qquad \frac{u_t}{u} = \phi_x(x,t)$$

gives solutions $u(x,t)$ to the original equation.

$$u(x,t) = Ce^{\int \phi dx}$$

Example (stationary boundary layer equations)

Consider the stationary boundary layer equations

$$\begin{aligned} uu_x + vu_y + \theta &= u_{yy} \\ u_x + v_y &= 0 \end{aligned} \tag{SBL}$$

where $\theta(x)$ is a given function.

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Generating differential invariants up to first order are

$$x \qquad u \qquad I = u_x + v_y \qquad J = u_y \qquad K = uu_x + vu_y$$

Our **automorphic system** takes the form

$$I = \omega(x, u) \quad J = \phi(x, u) \quad K = \psi(x, u)$$

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$$v = \frac{\psi - uu_x}{u_y} = \frac{\phi\phi_u - (\theta + uu_x)}{\phi},$$

which, when plugged into $u_x + v_y = \omega$ gives

$$u\phi_x = \phi^2\phi_{uu} + \theta\phi_u \qquad (*)$$

With a solution ϕ to (*) in hand, we find $u(x, y)$, $v(x, y)$ by solving

$$u_x + v_y = 0 \quad u_y = \phi \quad uu_x + vu_y = \phi\phi_u - \theta.$$

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Reason 2:
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Reason 3:
Moving frames may be used in the reconstruction process. The reconstruction process can be viewed as pushing the solution of the resolving equation off the cross-section, which can be done using a moving frame parametrized by the independent variables on the cross-section.

Review of moving framework

Notation

$\mathcal{D}$ the Lie pseudogroup of local diffeomorphisms on $M = \mathcal{X} \times \mathcal{U}$

$\mathcal{D}^{(n)}$ groupoid of n-jets of diffeomorphisms, source map σ, target map τ

Source coordinates x^i, u^α $\quad i = 1, \ldots, p$, $\alpha = 1, \ldots, q$

Target coordinates X^i, U^α

Jet coordinates $x^i, u^\alpha, X^i, U^\alpha, X^i_{x^j}, X^i_{u^\alpha}, U^\alpha_{x^i}, U^\alpha_{u^\beta}, \ldots$

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$\mathcal{G} \subset \mathcal{D}$ a Lie pseudo-group, $\quad \mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ the bundle of n-jets

Infinitesimal generator $\mathrm{v} = \xi^i \partial_{x^i} + \phi^\alpha \partial_{u^\alpha}$

Vector field coordinates satisfy *infinitesimal determining equations*

$$L^{(n)}(x, u, \xi^{(n)}, \phi^{(n)}) = 0$$

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The order 0 Maurer-Cartan forms are

$$\mu^{x^i} = dX^i - X^i_{x^j} dx^j - X^i_{u^\alpha} du^\alpha$$
$$\mu^{u^\alpha} = dU^\alpha - U^\alpha_{x^j} dx^j - U^\alpha_{u^\alpha} du^\alpha$$

and satisfy

$$\lambda(\xi^j) = \mu^{x^j} \qquad \lambda(\phi^\alpha) = \mu^{u^\alpha}$$

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Higher order M–C forms are obtained by Lie differentiation. The same linear determining equation is satisfied by the M–C forms

$$L^{(n)}(X, U, \mu^{(n)})$$

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Invariantization creates invariant functions/forms by lifting, then normalizing the group parameters (pulling back by the moving frame):

$$\iota(\omega) = \rho^* \lambda(\omega)$$

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Invariantization of the horizontal coframe dx^j defines the invariant "horizontal" coframe $\iota(dx^j) = \varpi^{x^j}$. The **invariant differential operators** $\mathcal{D}_{x^j}$ are defined as dual to the invariant coframe

$$dF \equiv \mathcal{D}_{x^j} F \varpi^{x^j},$$

where $\equiv$ denotes projection onto the invariant horizontal space.

Review of moving framework

The **recurrence relations** are the key to everything!

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For finite dimensional groups the recurrence relation is often written

$$d\iota(\omega) = \iota(d\omega) + \nu^{\kappa} \wedge \iota(\mathrm{v}_{\kappa}^{(\infty)}\omega)$$

where ν^{κ} are moving frame pull-backs of a basis of M–C forms dual to infinitesimal generators v_{κ}.

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The recurrence relations may be used, in conjunction with a choice of cross-section, to compute the structure of the algebra of differential invariants and moving frame pull-backs of Maurer-Cartan forms.

Example (nonlinear heat equation revisited)

Consider again the equation

$$u_t = u_{xx} - \frac{u_x^2}{u} \qquad \text{(NLH)}$$

and scaling symmetry $X = x \qquad T = t \qquad U = \lambda u.$

Use the cross-section $u = 1$ to find the moving frame:

$$\lambda u = 1 \qquad \Longrightarrow \qquad \rho = 1/u.$$

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Normalized invariants (up to second order)

$$x \qquad t \qquad I = \iota(u_x) = \frac{u_x}{u} \qquad J = \iota(u_t) = \frac{u_t}{u}$$

$$\iota(u_{xx}) = \frac{u_{xx}}{u} \qquad \iota(u_{xt}) = \frac{u_{xt}}{u} \qquad \iota(u_{tt}) = \frac{u_{tt}}{u}$$

x, t, I, J generate the algebra of differential invariants.

Example (nonlinear heat equation revisited)

Using the cross-section and the infinitesimal generator

$$\mathrm{v}^{(\infty)} = u\partial_u + u_x\partial_{u_x} + u_t\partial_{u_t} + u_{xx}\partial_{u_{xx}} + u_{xt}\partial_{u_{xt}} + u_{tt}\partial_{u_{tt}} + \cdots$$

we find the moving frame pullback $\nu = \rho^*(\mu)$:

$$0 = d\iota(u) = \iota(du) + \nu \wedge \iota(\mathrm{v}^{(\infty)}(u)) \implies \nu \equiv -I\varpi^x - J\varpi^t.$$

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Recurrence relations yield:

$$d\iota(x) = \iota(dx) + \nu \wedge \iota(\mathrm{v}^{(\infty)}(x)) \implies dx = \varpi^x$$

$$d\iota(t) = \iota(dt) + \nu \wedge \iota(\mathrm{v}^{(\infty)}(t)) \implies dt = \varpi^t$$

$$d\iota(u_x) = \iota(du_x) + \nu \wedge \iota(\mathrm{v}^{(\infty)}(u_x))$$

$$\implies dI \equiv [\iota(u_{xx}) - I^2]\varpi^x + [\iota(u_{xt}) - IJ]\varpi^t$$

$$d\iota(u_t) = \iota(du_t) + \nu \wedge \iota(\mathrm{v}^{(\infty)}(u_t))$$

$$\implies dJ \equiv [\iota(u_{xt}) - IJ]\varpi^x + [\iota(u_{tt}) - J^2]\varpi^t$$

Example (nonlinear heat equation revisited)

Thus (since $\mathcal{D}_x = D_x$ and $\mathcal{D}_t = D_t$),

$$D_x I = \iota(u_{xx}) - I^2 \qquad D_t I = \iota(u_{xt}) - IJ$$
$$D_x J = \iota(u_{xt}) - IJ \qquad D_t J = \iota(u_{tt}) - J^2$$

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The equation yields the additional "constrained syzygy"

$$\iota\left(u_t = u_{xx} - \frac{u_x^2}{u}\right) \implies J = \iota(u_{xx}) - I^2$$
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We arrive directly at the previous resolving equation

$$D_t I = D_x^2 I.$$

General description of the algorithm so far

Given: n^{th} order PDE $\Delta = 0$ with symmetry group or pseudogroup $\mathcal{G}$.

Step 1. Choose p independent invariants J^i to act as new independent variables. Add invariants K^α until the set $\{J^i, K^\alpha\}$ is a generating set of invariants. The K^α will act as new dependent variables.

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Step 3. Add the "constrained syzygy" from the original equation.

Result: A system of equations in K^α and their derivatives w.r.t. the J^i. These are the resolving equations to be solved.

Example (nonlinear wave equation)

Consider the equation

$$uu_{xy} - u_x u_y = u^3 \qquad u > 0 \qquad \text{(NLW)}$$

admitting the symmetry pseudo-group

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The cross-section $x = 0$, $u = 1$, $u_x = 0$, $u_{xx} = 0$, $\ldots$ yields normalized invariants

$$I = \iota(y) = y \qquad J = \iota(u_y) = \frac{u_y}{y}$$

$$K = \iota(u_{xy}) = \frac{u u_{xy} - u_x u_y}{u^3} \qquad L = \iota(u_{yy}) = \frac{u_{yy}}{u}$$

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We choose I, J as independent variables for the group foliation.

Example (nonlinear wave equation)

The universal recurrence relation $d\iota(\Omega) = \iota(d\Omega + \mathrm{v}^{(\infty)}\Omega)$ with the prolongation of the infinitesimal generator

$$\mathrm{v} = a(x)\partial_x - a'(x)u\partial_u$$

yields $\iota(a) = -\varpi^x,\ \iota(a_x) \equiv J\varpi^y,\ \iota(a_{xx}) \equiv K\varpi^y,$

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$$dI = \varpi^y \qquad dK \equiv \iota(u_{xxy})\varpi^x + [\iota(u_{xyy}) - 3JK]\varpi^y$$
$$dJ \equiv K\varpi^x + (L - J^2)\varpi^y \qquad dL \equiv \iota(u_{xyy})\varpi^x + [\iota(u_{yyy}) - LJ]\varpi^y$$

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The following syzygy is immediate

$$\mathcal{D}_y K = \mathcal{D}_x L - 3JK.$$

Invariantization of (NLW) yields the “constrained syzygy”

$$K = 1.$$

Example (nonlinear wave equation)

The chain rule yields

$$\mathcal{D}_x = \mathcal{D}_x I D_I + \mathcal{D}_x J D_J = K D_J$$
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The constrained syzyzgy $K = 1$ yields the **resolving equation**

$$D_J L = 3J.$$

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The chain rule yields

$$\mathcal{D}_x = \mathcal{D}_x I D_I + \mathcal{D}_x J D_J = K D_J$$
$$\mathcal{D}_y = \mathcal{D}_y I D_I + \mathcal{D}_y J D_J = D_I + (L - J^2) D_J$$

Our syzygy becomes

$$\mathcal{D}_y K = \mathcal{D}_x L - 3JK \implies D_I K + (L - J^2) D_J K = K D_J L - 3JK.$$

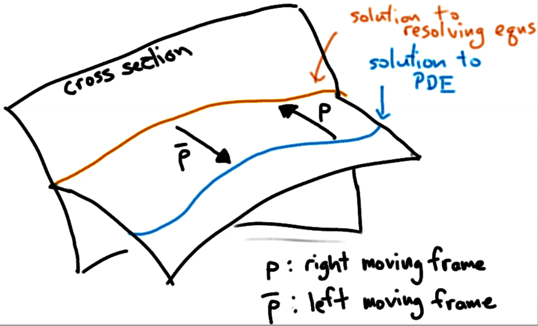
The constrained syzyzgy $K = 1$ yields the **resolving equation**

$$D_J L = 3J.$$

The solution is $L(I, J) = \frac{3}{2} J^2 + F(I)$, F an arbitrary function.

Reconstruction method

We view the resolving equations as PDE on the cross-section.
The *right moving frame* maps to the cross-section, the *left moving frame* maps away.



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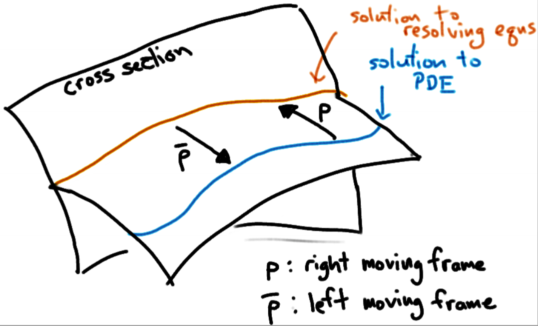

Goal: Find $\bar{\rho}(J^i)$, parametrized by J^i. Once $\bar{\rho}$ is found it can be applied to the solution of the resolving equations.

Method: Find a differential equation for $\bar{\rho}$ and solve it.

Reconstruction method

For example, for the equation

$$u_t = u_{xx} - \frac{u_x^2}{u} \qquad \text{(NLH)}$$

we found the resolving equation $D_t I = D_x^2 I$.

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$$u_t = u_{xx} - \frac{u_x^2}{u} \qquad \text{(NLH)}$$

we found the resolving equation $D_t I = D_x^2 I$.

The right moving frame $\rho(x, y)$ maps the solution to the cross-section:

$$\rho(x, y) \cdot (x, y, u, u_x, u_t) = (x, y, 1, I, J)$$

and the left moving frame maps back:

$$\bar{\rho}(x, y) \cdot (x, y, 1, I, J) = (x, y, u, u_x, u_t).$$

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To reconstruct a solution to (NLH) from a solution to the resolving equation, we compute $\bar{\rho}(x, y)$ and apply it to a solution of the resolving equation.

Reconstruction method

How to compute $\bar{\rho}$ *?*

Reconstruction method

How to compute $\bar{\rho}$?

One method for finite dimensional groups is to embed ρ in $GL(n)$ and use the trivial identity $\bar{\rho}\,\rho = Id$ to derive the relation

$$d\bar{\rho} = -\bar{\rho}\,(d\rho\,\rho^{-1}).$$

The expression $d\rho\,\rho^{-1}$ is a matrix of right Maurer–Cartan forms pulled back by the moving frame, and can be computed using the recurrence relation.

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Doesn't work for pseudogroups, and requires a representation of ρ.

Take away idea: write the differential of the left moving frame using the moving frame pull-backs of the right Maurer–Cartan forms.

Example (nonlinear heat equation reconstruction)

Recall the equation $u_t = u_{xx} - \dfrac{u_x^2}{u}$ with symmetry

$$X = x \qquad T = t \qquad U = \lambda u$$

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Right M–C form (others are zero):

$$\mu^u = dU - U_x dx - U_t dy - U_u du = dU - \lambda du$$

Right M–C form pullback (from the recurrence relation)

$$\rho^*(\mu^u) \equiv -I\varpi^x - J\varpi^t$$

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The group parameter $\bar{\lambda}$ for the left action satisfies

$$u = \frac{1}{\lambda} U = \bar{\lambda} U.$$

Left M–C form may be written

$$\mu^U = du - u_X dX - u_T dT - u_U dU = du - \bar{\lambda} dU$$

Example (nonlinear heat equation reconstruction)

Notice that

$$\mu^U = du - \bar{\lambda} dU = du - \frac{1}{\lambda} dU = -\frac{1}{\lambda}\big(dU - \lambda du\big)$$

and hence the left and right M–C forms are related by

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Thus

$$du - \bar{\lambda} dU = U d\bar{\lambda} = -\bar{\lambda}\mu^u$$

Pulling back by the right moving frame yields the equation for $\bar{\lambda}$:

$$d\bar{\lambda} \equiv I\varpi^x + J\varpi^t \equiv I dx + J dt \qquad (*)$$

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To perform reconstruction, we solve the resolving equations, plug the solution into (*) and find $\bar{\rho}$.

Example (nonlinear heat equation reconstruction)

Resolving equation is

$$D_t I = D_x^2 I$$

where $I = \iota(u_x)$, $J = \iota(u_t)$. Suppose $I(x,t)$ is a solution. Then

$$J = D_x I,$$

and the reconstruction equations are

$$D_x \bar{\lambda} = I\bar{\lambda} \qquad D_t \bar{\lambda} = \bar{\lambda} D_x I.$$

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A solution is $\bar{\lambda}(x,t) = e^{\int I(x,t)dx}$. Acting by this group element on the cross-section we obtain

$$(x,t,1) \mapsto (x,t,e^{\int I(x,t)dx}),$$

that is

$$u(x,t) = e^{\int I(x,t)dx}.$$

General strategy for reconstruction

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Step 5. Apply the reconstruction equation solution to the cross section.

Example (nonlinear wave equation reconstruction)

Recall the equation $uu_{xy} - u_x u_y = u^3$ admitting the pseudogroup

$$X = f(x) \qquad Y = y \qquad U = \frac{u}{f'(x)}.$$

Write the parameters for the left action as

$$x = g(X) \qquad y = Y \qquad u = \frac{U}{g'(X)}.$$

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We compute the left M–C form

$$\mu^X = dx - x_X dX - x_Y dY - x_U dU = dg - g_X dX,$$

which may be rewritten using the right M–C forms:

$$\mu^X = -g_X \mu^x$$

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Hence

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Together,

$$D_I g = D_J g \left(\frac{J^2 - L}{K} \right)$$

Example (nonlinear wave equation reconstruction)

Using the solution to the resolving equations

$$K = 1 \qquad L(I, J) = \frac{3}{2}J^2 + F(I)$$

the reconstruction equations become

$$\frac{\partial g}{\partial I} = -\frac{\partial g}{\partial J}\left(\frac{1}{2}J^2 + F(I)\right).$$

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If $g(I,J)$ is a solution to these equations, we then act on the cross-section to obtain a solution parametrized by I and J:

$$(x,t,u) = \left(g, I, \frac{1}{\frac{\partial g}{\partial J}}\right)$$

Example (Calogero nonlinear wave equation)

$$u_{xt} + u u_{xx} - F(u_x) = 0 \qquad \text{(CNLW)}$$

admits the symmetry pseudogroup

$$X = x + a(t) \qquad T = t \qquad U = u + a'(t)$$

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The prolonged infinitesimal generator is

$$\begin{aligned} \mathrm{v} = a\partial_x + a'\partial_u + (a'' - a'u_x)\partial_{u_t} \\ + (-a'u_{xx})\partial_{u_{xt}} + (a''' - a''u_x - 2a'u_{xt})\partial_{u_{tt}} + \cdots \end{aligned}$$

We choose the cross-section

$$x = 0 \qquad u = 0 \qquad u_t = 0 \qquad u_{tt} = 0 \qquad \cdots$$

Example (Calogero nonlinear wave equation)

Recurrence relation computations yield the normalizations

$$\iota(a) = -\varpi^x \qquad \iota(a_t) = -I_{10}\varpi^x \qquad \iota(a_{tt}) = -(I_{11} + I_{10}^2)\varpi^x$$

where $\iota(u_x) = I_{10}, \iota(u_{xt}) = I_{11}$, etc.

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$$dI_{10} \equiv I_{20}\varpi^x + I_{11}\varpi^t \qquad dI_{20} = I_{30}\varpi^x + I_{21}\varpi^t$$

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New *independent variables* $\quad t \qquad s = I_{10}$
New *dependent variables* $\quad K = I_{11} \qquad L = I_{20}$
Explicitly,

$$\mathcal{D}_x s = L \qquad \mathcal{D}_t s = K$$
$$\mathcal{D}_x K = I_{21} + sL \qquad \mathcal{D}_t K = I_{12}$$
$$\mathcal{D}_x L = I_{30} \qquad \mathcal{D}_t L = I_{21}$$

Example (Calogero nonlinear wave equation)

The invariant differential operators may be written

$$\mathcal{D}_x = L D_s \qquad \mathcal{D}_t = D_t + K D_s$$

corresponding to the dual relationship

$$\varpi^x \equiv \frac{1}{L} ds - \frac{K}{L} dt \qquad \varpi^t \equiv dt.$$

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The syzygies may be rewritten

$$\begin{aligned} D_t s &= K \\ L D_s K &= I_{21} + sL \qquad & D_t K + K D_s K &= I_{12} \\ L D_s L &= I_{30} \qquad & D_t L + K D_s L &= I_{21} \end{aligned}$$

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Comparing the I_{21} terms gives immediately:

$$L(K_s - s) = K L_s + L_t$$

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Invariantization of (CNLW) gives the constrained syzygy

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This must be solved (given a particular F).

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This must be solved (given a particular F).

We'll do this in a minute. First, we derive the the reconstruction equations.

Example (Calogero nonlinear wave equation)

Reconstruction. Find the left Maurer–Cartan forms:

$$\begin{aligned}
\mu^X &= dx - dX - b_T dT = db - b_T dT \\
\mu^T &= 0 \\
\mu^U &= du - dU - b_{TT} dT = db_T - b_{TT} dT
\end{aligned}$$

where b, b_T, b_{TT} are the pseudogroup parameters for the left action.

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where b, b_T, b_{TT} are the pseudogroup parameters for the left action. Use the relationship between the left and right M–C forms:

$$\begin{bmatrix} \mu^X \\ \mu^T \\ \mu^U \end{bmatrix} = - \begin{bmatrix} 1 & b_T & 0 \\ 0 & 1 & 0 \\ 0 & b_{TT} & 1 \end{bmatrix} \begin{bmatrix} \mu^x \\ \mu^t \\ \mu^u \end{bmatrix}$$

So,

$$\begin{aligned}
-\mu^x &= db - b_T dT \\
-\mu^u &= db_T - b_{TT} dT
\end{aligned}$$

Example (Calogero nonlinear wave equation)

Recall that we have found already the right moving frame pull-backs of the Maurer–Cartan forms:

$$\iota(a) = \rho^* \mu^x = -\varpi^x \equiv -\frac{1}{L} ds + \frac{K}{L} dt$$

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To determine equations for the left moving frame parameters, pull back by the right moving frame:

$$-\mu^x = db - b_T dT \implies db \equiv \frac{1}{L} ds + \left(b_T - \frac{K}{L} \right) dt$$

$$-\mu^u = db_T - b_{TT} dT \implies db_T \equiv \frac{s}{L} ds + \left(b_{TT} - \frac{sK}{L} \right) dt$$

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So the reconstruction equations (to second order) are

$$D_s b = \frac{1}{L} \quad D_t b = b_T - \frac{K}{L} \quad D_s b_T = \frac{s}{L} \quad D_t b_T = b_{TT} - \frac{sK}{L}$$

Example (Calogero nonlinear wave equation)

One can use the consequence for reconstruction:

$$D_s b = \frac{1}{L} \qquad D_s D_t b = -D_s\left(\frac{K}{L}\right) + \frac{s}{L}.$$

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Now, take $F(s) = \frac{s^2}{2}$, so (CNLW) is $\quad u_{xt} + u u_{xx} - \frac{u_x^2}{2} = 0.$

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The resolving equations

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simplify to

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The resolving equations

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simplify to

$$\frac{s^2}{2} L_s + L_t = 0.$$

Solving by method of characteristics gives

$$L(s,t) = H\left(\frac{s}{1 - \frac{1}{2}st}\right)$$

where H is an arbitrary function.

Example (Calogero nonlinear wave equation)

For simplicity, choose

$$L(s,t) = \frac{1}{\frac{1}{s} - \frac{1}{2}t}.$$

The corresponding reconstruction equations become

$$D_s b = \frac{1}{s} - \frac{1}{2}t \qquad D_s D_t b = \frac{1}{2}$$

A particular solution is

$$b(s,t) = \log s + \frac{1}{2}st.$$

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Using

$$D_t b = b_T - \frac{K}{L}$$

we find

$$b_T(s,t) = \frac{1}{4}ts^2 - s$$

Example (Calogero nonlinear wave equation)

Using these values for $b(s,t)$ and $b_T(s,t)$ we find the solution to (CNLW), parameterized by the invariants s,t:

$$(x,t,u) = (\log s + \frac{1}{2}st, t, \frac{1}{4}ts^2 - s).$$

Comments and Questions

- The entire algorithm may be viewed as extension of Mansfield's algorithm for integrating invariant ODE.

- A similar process may be used to find invariant, partially invariant, and differential invariant solutions.

- Reconstruction equations: what is their relation with the automorphic system?

- Reconstruction equations: what to do with higher order group parameters which appear at each level?

- Similarity with EDS algorithm of Anderson, Fels, Pohjanpelto.

- Application of symmetry techniques for solving resolving equations?