Darboux-Manakov-Zakharov Systems and Einstein Metrics

Peter Vassiliou

University of Canberra, Australia

Moving Frames in Geometry
Centre de Recherches Mathématiques,
Montreal, Canada
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Aims

Aims:

- Describe a remarkable class of involutive overdetermined linear PDE, Darboux-Manakov-Zakharov systems
- Review (a portion of) their role in geometry and integrable systems
- Main result
- Discuss possible role in constructing Einstein metrics
- Pose some questions

Some references

Dates are approximate, list incomplete:

- Darboux (1910),
- Zakharov-Manakov (1973,1985),
- Dubrovin, Novikov (1980-1988),
- Tsarev (1989-)
- V (1994)
- Kamran-Tenenblat (1996); *Got me started*,
- Zakharov (1998),
- Ferapontov (Numerous listed in arXiv)
- Anderson-Fels-V (2009)
- ...

Darboux-Manakov-Zakharov Systems

Overdetermined involutive linear systems

$$\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ji}(x)\frac{\partial u}{\partial x_i} - \Gamma_{ij}(x)\frac{\partial u}{\partial x_j} = 0, \quad 1 \leq i < j \leq n$$

called **Darboux-Manakov-Zakharov systems** (DMZ systems).

Sometimes we include an extra term

$$\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ji}(x)\frac{\partial u}{\partial x_i} - \Gamma_{ij}(x)\frac{\partial u}{\partial x_j} + C_{ij}u = 0, \quad 1 \leq i < j \leq n$$

Note: No summation over repeated indices!

A Theorem of Darboux

Theorem (Darboux)

Suppose overdetermined PDE system

$$\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ji}(x)\frac{\partial u}{\partial x_i} - \Gamma_{ij}(x)\frac{\partial u}{\partial x_j} = 0$$

for $u : \mathbb{R}^3 \to \mathbb{R}$ *is involutive. Then*

1. *there exist* $h_1, h_2, h_3 : \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$\Gamma_{ij} = \frac{1}{h_j}\frac{\partial h_j}{\partial x_i}$$

2. *functions*

$$\beta_{ij} = \frac{1}{h_i}\frac{\partial h_j}{\partial x_i} = \Gamma_{ij} \exp \int \left(\Gamma_{ij} dx_i - \Gamma_{ji} dx_j\right)$$

solve 2+1-dimensional, 3-wave resonant system.

3-wave resonant interaction system

The 2+1 dimensional, three-wave resonant interaction equation (3WRI): is the nonlinear PDE

$$\frac{\partial \beta_{jk}}{\partial x_i} = \beta_{ji}\beta_{ik}, \quad (i,j,k) \in \text{perm}(3,3)$$

arises in plasma physics, nonlinear optics & numerous other areas in physics (Zakharov-Manakov, $\approx$ 1973); figures prominently in discrete differential geometry.
It is “one half” of the Lamé equations for triply orthogonal coordinate systems

3WRI & Lamé equations

Turns out, 3WRI system forms a part of the Lamé equations for flat diagonal 3-metrics, *aka* triply orthogonal coördinate systems

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0, \quad (\longleftarrow \quad 3WRI)$$
$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{m\neq i,j} \beta_{mi}\beta_{mj} = 0.$$

—— The Lamé equations ——

Various 19^{th} *C* geometers contributed:

Bianchi, Darboux, Ribaucour, …

semi-Hamiltonian hydrodynamic type systems

A first order PDE system

$$\boxed{u^i_t = v^i(u^1, \ldots, u^n)u^i_x.} \tag{1}$$

is a strongly hyperbolic hydrodynamic type system if v^i are all distinct.

Equation (1) is semi-Hamiltonian if

$$\left(\partial_{u_i u_j} - \Gamma_{ji}\partial_{u_i} - \Gamma_{ij}\partial_{u_j}\right) U = 0, 1 \leq i < j \leq n,$$

is involutive where

$$\Gamma_{ij} = \frac{\partial_{u_j} v^i}{v^i - v^j}.$$

Arose from work of B. Dubrovin, S.P. Novikov, S. Tsarev ≈ 1989; numerous applications in analysis, geometry, continuum physics.

Ricci-diagonal metrics

Let H_i smooth real-valued functions on $\mathbb{R}^n$. Metric

$$g = \sum_{i=1}^{n} e^{2H_i}\, dx_i^2;$$

Define function

$$\beta_{jk} = \frac{\partial H_j}{\partial x_k} e^{H_j - H_k}.$$

Then, for $i \neq j$

$$\text{Ricci}_g(\boldsymbol{e}_i, \boldsymbol{e}_j) = -\left(\frac{\partial \beta_{ij}}{\partial x_k} - \beta_{ik}\beta_{kj}\right) e^{-H_i - H_j},\ k \neq i, j$$

A theorem of S. Tsarev

Theorem (S. Tsarev ≈ 1989)

Suppose a hydrodynamic system is semi-Hamiltonian. Then

1. *The (Tsarev) linear system*

$$\frac{\partial w^i}{\partial u_j}(u) = \Gamma_{ij}(u)(w^j(u) - w^i(u)) \qquad (2)$$

   is involutive

2. *If $w(u)$ solves (2), then $u^i_t = w^i(u)u^i_x$ is a commuting flow for the original system (1): $u^i_t = v^i(u)u^i_x$*

3. *Each semi-Hamiltonian system determines a diagonal and Ricci-diagonal metric; and conversely.*

Quotient Differential Systems

Definition

Let Ω be a Pfaffian system on smooth m'fold M, $\mu : G \times M \to M$ a free, regular Lie group action on M preserving Ω:

$$\mu(g)^*\Omega \subseteq \Omega, \quad \forall \quad g \in G.$$

Then the *quotient* of Ω by G is Pfaffian system Ω/G on M/G

$$\Omega/G = \left\{\omega \in \Lambda^1(M/G) \mid \pi^*\omega \in \Omega\right\},$$

where $\pi : M \to M/G$ natural projection.

View $(M/G,\ \Omega/G)$ as a *symmetry reduction* of $(M,\ \Omega)$.

Geometric construction of DMZ Systems

Let $\mathbb{J}$ denote products of jet spaces

$$\mathbb{J} = J^{k_1}(\mathbb{R},\mathbb{R}) \times J^{k_2}(\mathbb{R},\mathbb{R}) \times \cdots \times J^{k_r}(\mathbb{R},\mathbb{R})$$

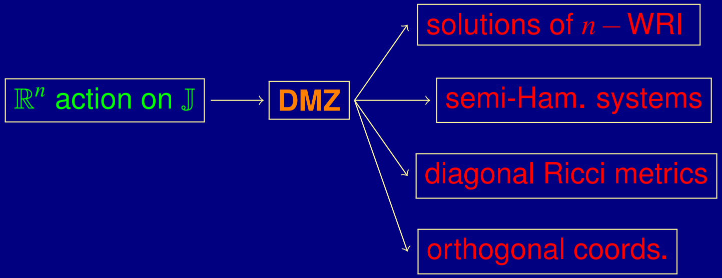
equipped with its *multi-contact distribution*, $\mathscr{C}$.

Theorem (Stud. App. Math., 2011)

$\mu : \mathbb{J} \times G \to \mathbb{J}$ be a free, regular action of $G = \mathbb{R}^q$ preserving $\mathscr{C}$.
Then $(\mathbb{J}/G, \mathscr{C}/G)$ embedds in $J^2(\mathbb{R}^n,\mathbb{R})$ as a DMZ system (involutive).

Call this the "geometric construction" of DMZ systems

Schematic Outline



where

$$\begin{aligned} \mathbb{J} &= \text{product of jet bundles, } J^\ell(\mathbb{R},\mathbb{R}) \\ \textbf{DMZ} &= \text{Darboux-Manakov-Zakharov linear system} \end{aligned}$$

Concrete Example: Setup
Group Actions

Example

Let $\mathbb{J}_1 = J^2(\mathbb{R},\mathbb{R}) \times J^2(\mathbb{R},\mathbb{R})$ with multi-contact distribution

$$\mathscr{C}_1 = \{\partial_x + x_1\partial_{x_0} + x_2\partial_{x_1}, \partial_{x_2}\} \oplus \{\partial_y + y_1\partial_{y_0} + y_2\partial_{y_1}, \partial_{y_2}\},$$

Define $\mathbb{R}^4$-action on $\mathbb{J}_1$:

$$\mu_1(\mathbf{t})(\boldsymbol{x},\boldsymbol{y}) = (x,\ x_0 + t_2 - t_1 x,\ x_1 - t_1,\ x_2,\ y,\ y_0 + t_4 - y t_3,\ y_2)$$

Concrete Example: Setup Group Actions

Similarly, let $\mathbb{J}_2 = J^4(\mathbb{R}, \mathbb{R})$ contact system

$$\mathscr{C}_2 = \{\partial_z + z_1\partial_{z_0} + z_2\partial_{z_1} + z_3\partial_{z_2} + z_4\partial_{z_3},\ \partial_{z_4}\}$$

and $\mathbb{R}^4$-action on $\mathbb{J}_2$:

$$\mu_2(\mathbf{t})(\boldsymbol{z}) = \mathrm{pr}^{(4)}\left(z,\ z_0 - \frac{z^2}{2}t_1 + \frac{z}{2}t_2 - \frac{1}{6}t_3 + \frac{z^3}{6}t_4\right)$$

Then let $\mathbb{J} = \mathbb{J}_1 \times \mathbb{J}_2$ and $\mathbb{R}^4$-action on $\mathbb{J}$

$$\mu(\mathbf{t})(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \mu_1(\mathbf{t})(\boldsymbol{x}, \boldsymbol{y}) \times \mu_2(\mathbf{t})(\boldsymbol{z})$$

From this data, one (very elementary) integration leads to the Darboux-Manakov-Zakharov system

Concrete Example: The DMZ System Constructed

$$\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ji}(x)\frac{\partial u}{\partial x_i} - \Gamma_{ij}(x)\frac{\partial u}{\partial x_j} = 0$$

where

$$\Gamma_{21} = \frac{z^3}{yz^3 - 1},$$

$$\Gamma_{31} = \frac{x + 2xyz^3 - yz^4 - 2z}{z(x-z)(yz^3-1)}, \quad \Gamma_{13} = \frac{z(1-yz^3)}{(x+2xyz^3-yz^4-2z)(x-z)},$$

$$\Gamma_{23} = \frac{z^3(2x-z)}{x+2xyz^3-yz^4-2z},$$

This overdetermined system is involutive (believe it or not)!
Note: Fels-Olver frames used here and in general theory!

The 3WRI solution from chosen group action

$$\begin{array}{c} J^2(\mathbb{R},\mathbb{R}) \times J^2(\mathbb{R},\mathbb{R}) \times J^4(\mathbb{R},\mathbb{R}) \\ \Big\downarrow \pi \\ \left(J^2(\mathbb{R},\mathbb{R}) \times J^2(\mathbb{R},\mathbb{R}) \times J^4(\mathbb{R},\mathbb{R})\right)/\mathbb{R}^4 \end{array}$$

From this, a solution of 3WRI can be constructed via 3 further quadratures

$$\boxed{\begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ \beta_{21} & 0 & \beta_{23} \\ \beta_{31} & \beta_{32} & 0 \end{pmatrix} = \frac{1}{x-z}\begin{pmatrix} 0 & 0 & z^2 \\ -z^2y^{-1} & 0 & z^3(2x-z)y^{-1} \\ -z^{-2} & 0 & 0 \end{pmatrix}}$$

Remarks

Moduli space of geometrically constructed DMZ systems?

Parametrised by any number of arbitrary functions of one variable $f^{\ell}(x_k)$.

Solvability of the Tsarev system?

For geometrically constructed semi-Hamiltonian systems commuting flows can be explicitly constructed.

In progress: DMZ systems & Einstein metrics

A Riemannian (or pseudo-Riemannian) metric g is Einstein if

$$\text{Ricci}_g = \lambda\, g.$$

It follows that any diagonal Einstein metric $g = \sum_i h_i^2\, dx_i^2$ gives rise to a DMZ system where the h_i are Lamé potentials.

Question: *Can one usefully characterise those DMZ systems which give rise to (diagonal) Einstein metrics?*

In progress: DMZ systems & Einstein metrics

Operators $\mathfrak{D}$ defined by

$$\mathscr{D}(u) = u_{x_i x_j} - \Gamma_{ji} u_{x_i} - \Gamma_{ij} u_{x_j} = 0, \quad 1 \leq i < j \leq n.$$

Each $\mathscr{D} \in \mathfrak{D}$ defines a family of diagonal metrics

$$\sum_i h_i^2 \, dx_i^2,$$

$$\Gamma_{ji} = \frac{\partial}{\partial x_j} \ln h_i.$$

Definition

Any such metric is said to be *associated* to DMZ operator $\mathscr{D}$.

DMZ & Einstein: some observations

Structure group of $\mathfrak{D}$?

- Class $\mathfrak{D}$ not invariant under general gauge transformations

$$\tilde{u} = T_\lambda\, u = \lambda(x)\, u.$$

However, is invariant under general "web transformations":

$$x_i \mapsto \xi_i(x_i),\ u \mapsto u.$$

- If λ satisfies $\mathscr{D}\lambda = 0$ then map $\mathscr{D} \to \lambda \cdot \mathscr{D} = \tilde{\mathscr{D}}$, induced from T_λ, valued in $\mathfrak{D}$.

Can questions about Einstein metrics (more generally Ricci diagonal metrics) be transfered to questions about the geometry of DMZ systems?

DMZ of Schwarzschild metric

Many known Einstein metrics of physical interest are warped products. E.g., Scharzschild metric is

$$g = -\sigma^2\, dt^2 + \frac{1}{\sigma^2}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

$$\sigma = \sqrt{1 - \frac{2m}{r}}.$$

Is this “product structure” apparent in its corresponding DMZ system?

DMZ of Schwarschild metric

Proposition

The DMZ system of the Schwarzschild metric g is gauge equivalent to the quotient of the linear second order hyperbolic PDE

$$\frac{\partial^2 u}{\partial t \partial r} + \frac{3m - r}{r(2m - r)} \frac{\partial u}{\partial t} = 0, \quad \frac{\partial^2 v}{\partial \theta \partial \varphi} - \cot\theta \frac{\partial v}{\partial \varphi} = 0$$

by the $\mathbb{R}$-action $(u, v) \mapsto (u + \varepsilon, v - \varepsilon)$.

DMZ of Schwarzschild metric

Note that each equation is web-equivalent to

$$\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \psi}{\partial y} = 0$$

and contact equivalent to the linear wave equation.

All diagonal Einstein metrics I have studied (from the “Exact Solutions” book) arise in a similar way. That is, they have the same web-geometry.

Summary

1. DMZ systems are implicated in a wide variety of problems in geometry and integrable systems;
2. A vast class of DMZ systems can be explicitly (and easily) constructed by symmetry reduction of jet spaces;
3. More generally, DMZ systems can be “glued” to produce new DMZ systems with prescribed properties
4. Some well known Einstein metrics arise via symmetry reductions of products of jets via correspondence in 2.

Further questions

- *Linear problem for diagonal Einstein.* Ricci diagonal metrics have DMZ as "linear problem".

  Question: Can this be refined to a linear problem for diagonal Einstein metrics?

- *Flat Lagrangian s'manifolds.* $\exists$ correspondence between flat Lagrangian submanifolds of $\mathbb{C}^n$ and solutions of the Lamé equations when the squared Lamé potentials $(h_1^2, \ldots, h_n^2)$ form a gradient field (Terng-Wang, 2008) - (flat) Egorov metrics. Geometrically constructed DMZ systems can often be deformed to solutions of the Lamé equations.

  Question: Can geometric construction of DMZ systems be controlled to give rise to flat Egorov metrics?

Further questions

- *Laplace transformations.* Multi-dimensional Laplace transformations preserve some geometric features but not all. E.g. Cartan s'manifolds are mapped to Cartan s'manifolds (Kamran-Tenenblat, 1996) but flatness is not in general preserved.

  Question: Are there s'manifold correspondences which preserve the Einstein property?
- *Orthogonal coordinates:* Many of the standard orthogonal coordinate systems are covered by my main thm.

  Question: Is this a useful link?

Thank you!