Symmetry Reduction of Exterior Differential Systems

Montreal 2011

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June 2011

Introduction Extension

Let $\mathbf{p} : M \to Q$ be a submersion and let $\mathcal{I} \subset \Omega^*(Q)$ be an EDS on M.

An EDS $\mathcal{E} \subset \Omega^*(M)$ on M with $\mathbf{p} * (\mathcal{I}) \subset \mathcal{E}$ is an ***extension*** of $\mathcal{I}$.
The EDS $\mathcal{E}$ is an ***integrable extension*** if,

1. There exists a Pfaffian system J on M such that
2. $\mathcal{E} = \langle \mathbf{p}^*(\mathcal{I}) + \mathcal{S}(J) \rangle_{\mathsf{alg}}$ and
3. J is transverse

$$\mathrm{ann}(J) \cap \ker(\mathbf{p}_*) = 0 \quad and$$

4. $\operatorname{rank} J = \dim N - \dim M$

If $\mathcal{I} =< \theta^i >_{\mathsf{alg}}$, then $\mathcal{E} =< \mathbf{p}^*(\theta^i), \zeta^a >_{\mathsf{alg}}$ where

$$d\zeta^a \equiv 0 \mod \{ \mathbf{p}^*(\theta^i), \zeta^a \}$$

- Property 3 implies given an immersed integral manifold $\sigma : S \to M$ for $\mathcal{E}$, then $\mathbf{p} \circ \sigma : S \to Q$ is an immersed integral manifold of $\mathcal{I}$.
- Property 2 implies given a solution $\sigma : S \to Q$ for $\mathcal{I}$, solutions to $\mathcal{E}$ can be obtained from σ using the Frobenius theorem.

Introduction Reduction

Let $\mathbf{p} : M \to Q$ be a smooth submersion and $\mathcal{I}$ an EDS on M.
The ***reduced differential system*** $\mathcal{I}/\mathbf{p}$ on Q is

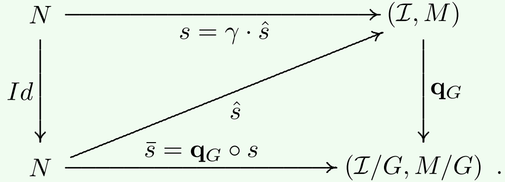
$$\mathcal{I}/\mathbf{p} = \{\, \theta \in \Omega^*(Q) \,|\, \mathbf{p}^*(\theta) \in \mathcal{I} \,\}.$$

If G is a Lie group which acts regularly on M with $\mathbf{q}_G : M \to M/G$ and by symmetries $g^*\mathcal{I} = \mathcal{I}$ then

$$\mathcal{I}/G = \{\, \theta \in \Omega^*(M/G) \,|\, \mathbf{q}_G^*(\theta) \in \mathcal{I} \,\}.$$

The basic idea for integral manifolds:

- If $s : N \to M$ is an integral manifold for $\mathcal{I}$ then $\bar{s} = \mathbf{q}_G \circ s$ is an integral manifold of $\mathcal{I}/G$, and
- If $\hat{s} : N \to M$ is ANY lift of $\bar{s} = \mathbf{q}_G \circ s$ then $s(x) = \gamma(x) \cdot \hat{s}(x)$ where $\gamma : N \to G$ satisfies a differential equation on G (Reconstruction)



Computing $\mathcal{I}/G$

Let Γ_G be the Lie algebra of infinitesimal generator.
Let $\mathbf{\Gamma}_G = \ker \mathbf{q}_{G*}$ be point-wise span of elements in Γ_G (Vertical space)

Let $\mathcal{I}_{sb} = \{ \ \theta \in \mathcal{I} \mid X \lrcorner \theta = 0 \text{ for all } X \in \Gamma_G \}$

Theorem: The subset $\mathcal{I}_{sb}$ is G-invariant and

$$\langle \mathcal{I}_{sb} \rangle = \langle \ \mathbf{q}_G^*(\mathcal{I}/G) \rangle.$$

Proof: Clearly we have

$$\mathcal{I}_{sb}/G \subset \mathcal{I}/G \quad \text{a}nd \quad \mathbf{p}^*(\mathcal{I}/G) \subset \mathcal{I}_{sb}$$

therefore $\mathcal{I}/G = \mathcal{I}_{sb}/G$.

Note that $\mathbf{\Gamma}_G$ are a subset of the Cauchy characteristics for $\langle \mathcal{I}_{sb} \rangle$. If G has connected orbits then $\mathcal{I}/G$ is Cauchy reduction of $\langle \mathcal{I}_{sb} \rangle$ by $\mathbf{\Gamma}_G$.

Theorem 2: If $\theta \in \mathcal{I}_{sb}$ and $\sigma : M/G \to M$ is a local cross-section then $\sigma^* \theta \in \mathcal{I}/G$.

Example 1: $J^3(\mathbf{R},\mathbf{R}) = (x, v, v_x, v_{xx}, v_{xxx})$

$$\mathcal{I} = \langle dv - v_x dx, dv_x - v_{xx}, dv_{xx} - v_{xxx} dx, dv_{xxx} \wedge dx \rangle$$

$$\Gamma_G = \text{span}\{\ X_1 = \partial_v,\ X_2 = pr(v\partial_v),\ X_3 = pr(v^2\partial_v)\}$$

Compute semi-basic one-forms $\theta(X_i) = 0$ for all $X_i \in \Gamma$

$$\det(\theta^i(X_j)) = 2u_x^3$$

Therefore on $M = \{u_x \neq 0\}$,

$$\mathcal{I}^1_{sb} = \{0\}.$$

We have $\mathcal{I}^2 = \Omega^2(M)$, and the differential invariant is the Schwartzian

$$\kappa = \frac{v_{xxx}}{v_x} - \frac{3v_{xx}^2}{v_x^2}, \quad \text{and}$$

$$\mathcal{I}^2_{sb} = \langle d\kappa \wedge dx \rangle.$$

Therefore on $M/\Gamma = (x, \kappa)$

$$\mathcal{I}/\Gamma = \text{span}\{d\kappa \wedge dx\}$$

The reconstruction problem is finding a curve in $\mathbf{R}\mathrm{P}^1$ with prescribed Schwartzian $\kappa = f(x)$.

Pfaffian Systems and Transversality

From now on $\mathcal{I}$ is Pfaffian system: $I \subset T^*M$, and $\mathcal{I} = \langle \mathcal{S}(I) \rangle_{\mathrm{diff}}$
The group acts ***transversally*** to $\mathcal{I}$ (or I) if

$$\mathrm{ann}(I) \cap \mathbf{\Gamma}_G = 0$$

For simplicity assume G acts freely: $\operatorname{rank} \ker \mathbf{q}_{G*} = \dim G$
Quotients/Solutions behave well for Transverse actions.

1. If $s : N \to M$ is an (immersed) integral manifold of $\mathcal{I}$ then $\mathbf{q}_G \circ s : N \to M/G$ is an immersed integral manifold of $\mathcal{I}/G$ and the symmetry group of $s(N) \subset M$ is at most discrete
2. $\operatorname{rank} I_{sb} = \operatorname{rank} I - \operatorname{rank} \mathbf{\Gamma}_G = \operatorname{rank} I - \dim G$
3. There exists $J \subset T^*M$; $\operatorname{rank} J = \operatorname{rank} \ker \mathbf{q}_{G*}$; transverse to $\mathbf{q}_G$ so that

$$\mathcal{I} = \mathcal{S}(J) + \mathbf{q}_G^*(\mathcal{I}/G).$$

That is $\mathcal{I}$ is an integrable extension of $\mathcal{I}/G$.

Rank I = Dim G

[i] G acts freely on M ($\mathbf{q}_G : M \to M/G$ is a left PFB)
[ii] Pfaffian system $I \subset T^*M$, G is symmetry.
[iii] rank $I = \dim G$,
[iv] G acts transversally $\mathrm{ann}(I) \cap \Gamma_G = 0$.

This implies

[iii], [i] $\mathrm{rank}\,\mathrm{ann}(I) = \dim M - \dim G = \dim M/G$
[iv] $\mathrm{ann}(I)$ is horizontal
[ii] $g_* \mathrm{ann}(I) = I$ for all $g \in G$

Therefore $\mathrm{ann}(I)$ is a connection on $M \to M/G$

$$\mathcal{I} = \langle \omega, \Omega \rangle; \quad \mathcal{I}/G = \langle \mathbf{R} \rangle$$

where ω is the connection form, $\mathbf{R}$ the curvature tensor.
Finding integral curves for $\mathcal{I}$ are horizontal lifts of curves $\bar{s} : \mathbf{R} \to M/G$:

$$\hat{s} : \mathbf{R} \to M;\; s(t) = \gamma(t) \cdot \hat{s}(t);\; \gamma : \mathbf{R} \to G$$

$$\frac{d\gamma}{dt} = -L_{\gamma *}\omega\Big(\frac{d\hat{s}}{dt}\Big) \quad \text{(Lie type).}$$

Example 1 was of this type: $G = SL(2, \mathbf{R})$.

Example ODE: $u_{xxx} = 3u_{xx}^2 u_x^{-1} + u_{xx}^3 u_x^{-5}$

$$M = (x, u, u_x, u_{xx})\,;\; I = \mathrm{span}\{du - u_x dx, du_x - u_{xx} dx, du_{xx} - \left(\frac{3u_{xx}^2}{u_x} + \frac{u_{xx}^3}{u_x^5}\right) dx\}$$

is invariant under the solvable $G = (a, b, c)$

$$x' = x + a + cu,\ u' = u + b,\ u'_{x'} = (1 + cu_x)^{-1} u_x,\ u'_{x'x'} = (1 + cu_x)^{-3} u_{xx}.$$

Choose cross-section $\hat{s}(t) = (0, 0, 1, t)$ of (x, u, u_x, u_{xx}); Write

$$s(t) = \gamma(t) \cdot \hat{s}(t) = (x = a(t),\ u(t) = b(t), u_x = (1 + c(t)), u_{xx} = (1 + c(t))^{-3} t$$

The equations of Lie type for $\gamma : \mathbf{R} \to G$ are

$$\dot{a} = (1 + c)t^{-3},\ \dot{b} = t^{-3},\ \dot{c} = -t^{-2}.$$

Integrate by quadratures (G is solvable).

- $I \subset T^*M$ is co-dimension 1 for an n-th order ODE
- G an n-dimensional symmetry group acting freely and transversally
- The connection $\mathrm{ann}(I)$ is one-dimensional (flat).
- Choose a cross-section $\hat{s} : \mathbf{R} \to M$ and make horizontal $s = \gamma(t) \cdot \hat{s}(t)$
- Solve equation of Lie type for $\gamma(t)$.

Free and Transverse: $\dim G \leq \operatorname{rank} I$

– Let Z_i, $1 \leq i \leq r$ be a basis of right invariant vector fields, and let τ^i be the dual right invariant one-forms to Z_i on G. Let $[\, Z_i,\, Z_j\,] = C^k_{ij} Z_k$.

– Let $\Gamma_G = \operatorname{span}\{X_1, X_2, \dots X_r\}$ corresponding to the choice of Z_i.

Theorem About each point $x \in M$ there exists a G-invariant open set U and a co-frame $\{\, \theta^i, \bar\eta^a, \bar\sigma^\alpha \,\}$ on U where

$$1 \leq i \leq \dim G,\ 1 \leq a \leq \operatorname{rank} I - \dim G,\ 1 \leq \alpha \leq \dim M/G$$

such that
[i] $\mathcal{I} = \langle \theta^i, \bar\eta^a \rangle_{\text{diff}}$;
[ii] the forms $\bar\eta^a$, $\bar\sigma^\alpha$ are G basic;
[iii] the forms θ^i satisfy $\theta^i(X_j) = \delta^i_j$,
[iv] the structure equations are

$$d\bar\sigma^\alpha = 0,\ d\bar\eta^a \equiv 0 \bmod \{\, \bar\sigma^\alpha,\, \bar\eta^a \,\} \quad \text{and}$$

$$d\theta^i = \bar A^i_{\alpha\beta}\, \bar\sigma^\alpha \wedge \bar\sigma^\beta + B^i_{a\beta} \bar\eta^a \wedge \bar\sigma^\beta - \frac{1}{2} C^i_{jk}\, \theta^j \wedge \theta^k.$$

[v] The reduced EDS is $\mathcal{I}/G|_{\overline{U}} = \langle \bar\eta^a, d\bar\eta^a, \bar A^i_{\alpha\beta}\, \bar\sigma^\alpha \wedge \bar\sigma^\beta \rangle$

$$
\begin{aligned}
\mathcal{I} &= \langle \theta^i, \bar{\eta}^a, d\bar{\eta}^a, \bar{A}^i_{\alpha\beta}\, \bar{\sigma}^\alpha \wedge \bar{\sigma}^\beta \rangle \\
\mathcal{I}/G &= \langle \bar{\eta}^a, d\bar{\eta}^a, \bar{A}^i_{\alpha\beta}\, \bar{\sigma}^\alpha \wedge \bar{\sigma}^\beta \rangle \\
d\theta^i &= -\frac{1}{2} C^i_{jk}\, \theta^j \wedge \theta^k + \bar{A}^i_{\alpha\beta}\, \bar{\sigma}^\alpha \wedge \bar{\sigma}^\beta + B^i_{a\beta} \bar{\eta}^a \wedge \bar{\sigma}^\beta
\end{aligned}
$$

Remark: If $\dim G = \operatorname{rank} I$ then no $\bar{\eta}^a$ terms.

Corollary 1: $\mathcal{I} = \{\theta^i\} + \mathbf{q}^*_G(\mathcal{I}/G)$ and is an integrable extension of $\mathcal{I}/G$.

Corollary 2: Let $s : N \to M$ be an integral manifold of $\mathcal{I}$, then $\bar{s} = \mathbf{q}_G \circ s$ is an integral manifold of $\mathcal{I}/G$. If $\hat{s}$ a lift of $\bar{s}$. There exists a unique function $\gamma : N \to G$ such that

$$s(x) = \gamma(x) \cdot \hat{s}(x)$$

and γ satisfies an equation of generalized Lie type.

Corollary 3: Let $s_1, s_2 : N \to M$ be integral manifolds. If there exists $g \in G$ such that $s_2 = g \circ s_1$ then $\mathbf{q}_G \circ s_2 = \mathbf{q}_G \circ s_1$. The converse holds locally.

Proof

The forms $\bar{\eta}^a$ are G-invariant sections of I_{sb} which has rank $\operatorname{rank} I - \dim G$.
The closed forms $\bar{\sigma}^\alpha$ are a complement to $\bar{\eta}^a$ so that $T^*(M/G) = \operatorname{span} \eta^a, \sigma^\alpha$.
To see where the forms θ^i come from, choose a local trivialization with open set U and map $(\mathbf{q}_G, \phi) : U \to \overline{U} \times G$
Let $\omega^i = \phi^*(\tau^i)$, where τ^i are the right-invariant forms on G defined above and dual to Z_i. Note that $\omega^i(X_j) = \delta^i_j$, and
Now by transversality there exists smooth function S^i_a, T^i_α on U so that

$$\omega^i + S^i_a \bar{\eta}^a + T^i_\alpha \bar{\sigma}^\alpha \in \mathcal{I},$$

Since $\eta^a \in \mathcal{I}$ let

$$\theta^i = \omega^i + T^i_\alpha \bar{\sigma}^\alpha,$$

At this point with $\{\, \theta^i, \bar{\eta}^a, \bar{\sigma}^\alpha \,\}$, parts **[i]**, **[ii]**, **[iii]** of the theorem hold.
By direct calculation using the Maurer-Cartan equations for ω^i, the structure equations for $d\theta^i$ are obtained.

Example 3: $J^2(\mathbf{R}, \mathbf{R}^2)$ **with coordinates** $(t, x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y})$

$$I = \langle dx - \dot{x}dt, dy - \dot{y}dy, d\dot{x} - \ddot{x}dt, d\dot{y} - \ddot{y}dt, d\ddot{x} \wedge dt,\ d\ddot{y} \wedge dt \rangle$$

The prolongation of $SE(2)$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

is a symmetry group of I (by prolongation) which is free and transverse on $M = \{(\dot{x}, \dot{y}) \neq (0,0)\} \subset J^2(\mathbf{R}, \mathbf{R})$. (Free on J^1, not transverse)

We have rank $I = 4$, therefore 1-semi-basic one form and

$$\mathcal{I}_{sb} = \langle \bar{\eta} = \dot{x}d\dot{x} + \dot{y}d\dot{y} - (\dot{x}\ddot{x} + \dot{y}\ddot{y})dt, d\bar{\eta}, \left(\dot{y}d\ddot{x} - \dot{x}d\ddot{y} + \frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{\dot{x}^2 + \dot{y}^2}(\dot{x}d\dot{y} - \dot{y}d\dot{x}) \right) \wedge dt \rangle$$

Let $\sigma : \bar{M} \to M$ be the cross-section,

$$(t, x = 0, y = 0, \dot{x} = 0, \dot{y} = v, \ddot{x} = k_1, \ddot{y} = k_2).$$

The reduced EDS is (pullback semi-basic forms)

$$\mathcal{I}/G = \ \langle dv - k_2 dt, dk_2 \wedge dt, dk_1 \wedge dt \rangle$$

Note : k_1 is the planar curvature.

A typical integral manifold for

$$\mathcal{I}/G = \langle dv - k_2 dt, dk_2 \wedge dt, dk_2 \wedge dt \rangle$$

is

$$\bar{s}(t) = (t, v = v(t), k_2 = \frac{dv}{dt}, k_1 = k(t)), \ v(t) \neq 0.$$

An integral manifold in M which projects to $\bar{s}$ is of the form [1]

$$s(t) = \mu(A(t), \sigma \circ \bar{s}(t))$$

where $\sigma \circ \bar{s}$ is the lift and $A : \mathbf{R} \to SE(2)$ satisfies

$$\begin{aligned} \frac{da}{dt} &= -v(t) \sin\theta(t) = 0, \ \frac{db}{dt} = v(t)\cos\theta(t), \\ \frac{d\theta}{dt} &= -\frac{k(t)}{v(t)}. \end{aligned}$$

An equation of Lie type for $\alpha : \mathbf{R} \to \mathbf{se}(2)$,

$$\alpha(t) = \left(0, -v(t), \frac{k_1(t)}{v(t)}\right).$$

[1] Again finding an integral manifold to $\mathcal{I}$ projecting to $\bar{s}$ is the prescribed "curvature" problem

Example 1 Revisited:

$M_1 = J^3(\mathbf{R},\mathbf{R}) = (x, v, v_x, v_{xx}, v_{xxx})$, $I_1 = \text{span}\{dv - v_x dx, dv_x - v_{xx}, dv_{xx} - v_{xxx}\}$

$\Gamma_G = \text{span}\{\partial_v, pr(v\partial_v), pr(v^2\partial_v)\}$

Take a second copy

$$M_2 = J^3(\mathbf{R},\mathbf{R}) = (y, w, w_y, w_{yy}, w_{yyy})\ ,\ I_2 = \langle \theta^w, \theta^w_y, \theta^w_{yy} \rangle$$

and on $M_1 \times M_2$ let

$$\Gamma_{G_{diag}} = \text{span}\{\partial_v - \partial_w, pr(v\partial_v + w\partial_w), pr(v^2\partial_v - w^2\partial_w)\}$$

We compute $(\mathcal{I}_1 + \mathcal{I}_2)/\Gamma_{G_{diag}}$ using the frame with $\theta^i_1(X_j) = \theta^2(Y_j) = \delta^i_j$:

$$\mathcal{I}_{sb} = \{\theta^i_1 - \theta^i_2, di_1 \wedge dx, di_2 \wedge dy\}$$

The joint differential invariants are:

$$u = \log\frac{2v_x w_y}{(v+w)^2},\ u_x = D_x(u) = \frac{v_{xx}}{v_x} - 2\frac{v_x}{v+w}, u_y = D_y(u), \ldots$$

The quotient $\mathcal{I}/\Gamma_{G_{diag}}$ is the standard EDS for $u_{xy} = e^u$,

$\langle du - u_x dx - u_y dy, du_x - u_{xx} dx - e^u dy, du_y - e^u dx - u_{yy} dy, d\hat{i}_1 \wedge dx, d\check{i}_2 \wedge dy \rangle$

where $u_x = D_x(u), u_y = D_y(u)$, etc. are coordinates on M/Γ_{G_diag}. While

$$\hat{i}_1 = v_{xxx} v_x^{-1} - 3v_{xx}^2 v_x{}^{-2} = u_{xx} - u_x^2/2$$

Bäcklund Transformations: An Application

Two exterior differential systems (EDS) $\mathcal{I}_1 \subset \Omega^*(M_1)$ and $\mathcal{I}_2 \subset \Omega^*(M_2)$ on manifolds M_1 and M_2 are related by a ***Bäcklund transformation***, if there exists an EDS $\mathcal{B} \subset \Omega^*(N)$ on a manifold N which is ***simultaneously*** an integrable extension for both $\mathcal{I}_1$ and $\mathcal{I}_2$,

$$\begin{array}{ccc} & (\mathcal{B}, N) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ (\mathcal{I}_1, M_1) & & (\mathcal{I}_2, M_2) \end{array} \; .$$

We call $\mathcal{B}$ the Bäcklund transformation.

1- Take a solution $\sigma : S \to M_1$ of $\mathcal{I}_1$.

2- Build a solution $\hat{\sigma} : S \to N$ of $\mathcal{B}$ by the Frobenius Theorem.

3- Project to a solution of $\mathcal{I}_2$.

4- Go the other way starting with $\sigma : S \to M_2$.

Commutative Diagrams

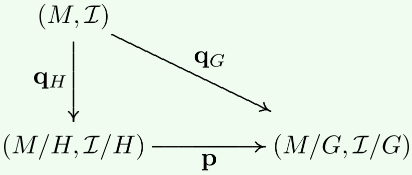
Let G act on M as symmetries for $\mathcal{I}$,

Let H be a sub-group

Let $\mathbf{p} : M/H \to M/G$ be the orbit map

$$\mathbf{p}(Hx) = Gx \quad x \in M$$

Theorem : The orbit mapping $\mathbf{p} : M/H \to M/G$ is a surjective submersion which gives rise to the following commutative diagram of EDS



If G is transverse to I then the diagram is a commutative diagram of integrable extensions.

Note: $\mathcal{I}/H/\mathbf{p} = \{\theta \in \Omega^*(M/G) \mid \mathbf{p}^*\theta \in \mathcal{I}/H\ \}$. This is NOT symmetry reduction.

Bäcklund Transformations by Symmetry Reduction

Using the previous diagram can construct Bäcklund transformations:

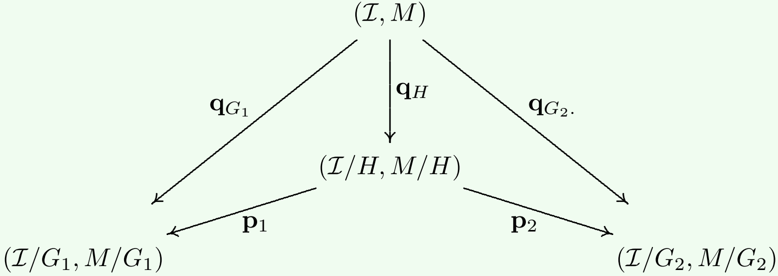
Let $\mathcal{I}$ be a Pfaffian system on M

Let G_1 and G_2 be symmetry groups acting transversally to I.

Let H be a common subgroup of G_1 and G_2

Let $\mathbf{q}_H : M \to M/H, \quad \mathbf{q}_{G_i} : M \to M/G_i, \quad \mathbf{p}_i : M/H \to M/G_i.$

Theorem A: The diagram



is a commutative diagram of integrable extensions.

Therefore

$$\begin{array}{ccc} & (M/H, \mathcal{I}/H) & \\ \swarrow \mathbf{p}_1 & & \mathbf{p}_2 \searrow \\ (M/G_1, \mathcal{I}/G_1) & & (M/G_2, \mathcal{I}/G_2) \end{array}$$

is a Bäcklund transformation between $\mathcal{I}/G_1$ and $\mathcal{I}/G_2$.

If the actions are free then the fibre dimensions are

$$\dim G_1 - \dim H \quad and \quad \dim G_2 - \dim H$$

Example: Bäcklund Transformation by Symmetry Reduction

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_x, v_{xx}, v_{xxx}, y, w, w_y, w_{yy}, w_{yyy}).$$

$$\begin{aligned}\mathcal{I} = \ &\langle dv - v_x\,dx,\ dv_x - v_{xx}\,dx,\ dv_{xx} - dv_{xxx}\,dx,\ dv_{xxx}\wedge dx,\\ &dw - w_y\,dy,\ dw_y - w_{yy}\,dy,\ dw_{yy} - w_{yyy}\,dy,\ dw_{yyy}\wedge dy\rangle\end{aligned}$$

Symmetries of $\mathcal{I}$ (infinitesimal)

$$X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w), X_3 = \partial_v + \partial_w, X_4 = pr(v^2\partial_v - w^2\partial_w)$$

Let

$$\begin{aligned}\Gamma_{G_1} = &\quad \{X_1, X_2, X_3\} &&= \{\partial_v, \partial_w, pr(v\partial_v + w\partial_w)\}\\ \Gamma_{G_2} = &\quad \{X_1, X_2, X_4\} &&= \{\partial_v - \partial_w, pr(v\,\partial_v + w\,\partial_w), pr(v^2\partial_v - w^2\partial_w)\}\\ \Gamma_H = &\quad \Gamma_{G_1}\cap\Gamma_{G_2} = \{X_1, X_2\} &&= \{\partial_v - \partial_w, pr(v\,\partial_v + w\,\partial_w)\}\end{aligned}$$

and construct the commutative diagram.
Note- The fibres of $\mathbf{p}_a$ are dimension 1 ($\dim\Gamma_{G_a} - \dim H$).

Example Part 3: The $\Gamma_H = \{X_1, X_2\}$ reduction

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_x, v_{xx}, v_{xxx}, y, w, w_y, w_{yy}, w_{yyy}).$$

$$\mathcal{I} = \langle dv - v_x\,dx,\ dv_x - v_{xx}\,dx,\ dv_{xx} - dv_{xxx}\,dx,\ dv_{xxx} \wedge dx,$$
$$dw - w_y\,dy,\ dw_y - w_{yy}\,dy,\ dw_{yy} - w_{yyy}\,dy,\ dw_{yyy} \wedge dy\rangle$$

Γ_H	$\{X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w)\}$
Invariants	$V = \log \dfrac{v_x}{v+w},\ W = \log \dfrac{w_y}{v+w}$

$$\boxed{J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R})}$$

$$\Big\downarrow \{X_1, X_2\}$$

$$\boxed{V_y = -e^W, W_x = -e^V}$$

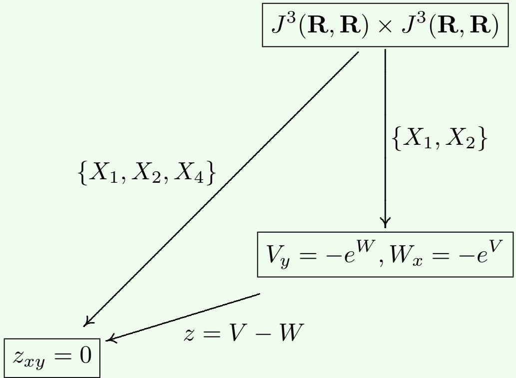
Example Part 3: The $\Gamma_{G_1} = \{X_1, X_2, X_3\}$ reduction

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_x, v_{xx}, v_{xxx}, y, w, w_y, w_{yy}, w_{yyy}).$$

$$\mathcal{I} = \langle dv - v_x\,dx,\ dv_x - v_{xx}\,dx,\ dv_{xx} - dv_{xxx}\,dx,\ dv_{xxx} \wedge dx,$$
$$dw - w_y\,dy,\ dw_y - w_{yy}\,dy,\ dw_{yy} - w_{yyy}\,dy,\ dw_{yyy} \wedge dy\rangle$$

Γ_{G_1} $\quad \{X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w), X_3 = \partial_v + \partial_w\}$

Invariant $\quad z = \log v_x - \log w_y, (\Gamma_H : V = \log \dfrac{v_x}{v+w},\ W = \log \dfrac{w_y}{v+w})$

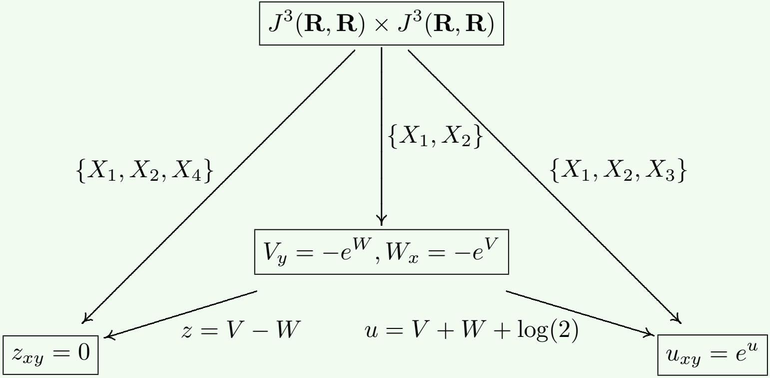


Example Part 3: The $\Gamma_{G_2} = \{X_1, X_2, X_4\}$ reduction

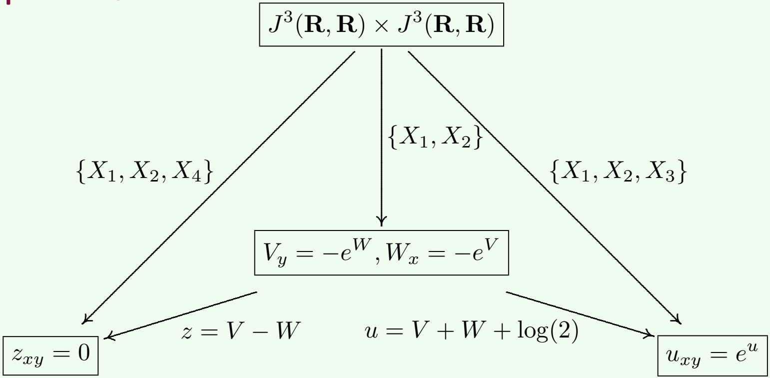
$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_x, v_{xx}, v_{xxx}, y, w, w_y, w_{yy}, w_{yyy}).$$

$$\mathcal{I} = \langle dv - v_x\,dx,\ dv_x - v_{xx}\,dx,\ dv_{xx} - dv_{xxx}\,dx,\ dv_{xxx} \wedge dx,$$
$$dw - w_y\,dy,\ dw_y - w_{yy}\,dy,\ dw_{yy} - w_{yyy}\,dy,\ dw_{yyy} \wedge dy\rangle$$

Γ_{G_1}	$\{X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w), X_4 = pr(v^2\partial_v + w^2\partial_w)\}$
Invariant	$u = \log v_x + \log w_y - 2\log(v+w) + \log 2$



Example Part 3:



$$z = V - W, \qquad z_x = V_x - W_x, \quad z_y = V_y - W_y$$

$$u = V + W + \log 2, \qquad u_x = V_x + W_x, \quad u_y = V_y + W_y$$

Writing $W_x = e^{-V}$ and $V_y = e^{-W}$ gives

$$z_x - u_x = \sqrt{2}\, e^{\frac{z+u}{2}}, \quad z_y + u_y = -\sqrt{2}\, e^{\frac{u-z}{2}}$$

Example Part 3: Deprolongation

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_x, v_{xx}, v_{xxx}, y, w, w_y, w_{yy}, w_{yyy}).$$

$$\mathcal{I} = \langle dv - v_x\, dx,\ dv_x - v_{xx}\, dx,\ dv_{xx} - dv_{xxx}\, dx,\ dv_{xxx} \wedge dx,$$
$$dw - w_y\, dy,\ dw_y - w_{yy}\, dy,\ dw_{yy} - w_{yyy}\, dy,\ dw_{yyy} \wedge dy\rangle$$

$$X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w), X_3 = \partial_v + \partial_w, X_4 = pr(v^2\partial_v - w^2\partial_w)$$

$$\Gamma_{G_1} = \{X_1, X_2, X_3\},\ \Gamma_{G_2} = \{X_1, X_2, X_4\},\ \Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \{X_1, X_2\}.$$

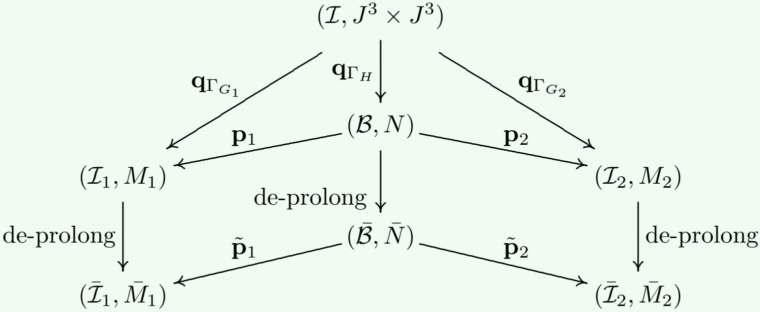
The quotients are:

$\mathcal{I}/\Gamma_{G_a}$ are rank 3 Pfaffian systems on the 7 manifolds M/Γ_{G_a} ($\mathbf{R}^{10} - \dim G_a$)

$\mathcal{I}/\Gamma_H$ is a rank 4 Pfaffian system on the 8 manifold M/Γ_H ($\mathbf{R}^{10} - \dim H$)

Our diagram can be de-prolonged.

Example Part 3: Deprolongation of Diagram



– $\bar{\mathcal{I}}_a$ be the Monge-Ampère representations on $z_{xy} = 0$ and $u_{xy} = e^u$ on five manifolds $\bar{M}_a$.

– $\bar{\mathcal{B}}$ is a rank 2 Pfaffian system on a six manifold $\bar{N}$.

Example Part 3:Deprolongation of $(\mathcal{I}, J^3 \times J^3)$

Instead of starting with $\mathcal{I}$ on $J^3 \times J^3$ start with $J^2 \times J^2$:

$$\tilde{M} = J^2(\mathbf{R},\mathbf{R}) \times J^2(\mathbf{R},\mathbf{R}) = \mathbf{R}^8(x, v, v_x, v_{xx}, y, w, w_y, w_{yy}).$$

$$\mathcal{I} = \langle dv - v_x\, dx,\ dv_x - v_{xx}\, dx,\ dv_{xx} \wedge dx,$$
$$dw - w_y\, dy,\ dw_y - w_{yy}\, dy,\ dw_{yy} \wedge dy\rangle$$

$$X_1 = \partial_v - \partial_w, X_2 = pr(v\,\partial_v + w\,\partial_w), X_3 = \partial_v + \partial_w, X_4 = pr(v^2\partial_v - w^2\partial_w)$$

$$(\mathcal{C}_1 + \mathcal{C}_2, J^2 \times J^2)$$

$$\mathbf{q}_{\tilde{\Gamma}_{G_1}} \qquad \mathbf{q}_{\tilde{\Gamma}_H} \qquad \mathbf{q}_{\tilde{\Gamma}_{G_2}}.$$

$$\bar{\mathcal{B}} = (\mathcal{C}_1 + \mathcal{C}_2)/\tilde{\Gamma}_H$$

$$\tilde{\mathbf{p}}_1 \qquad \mathbf{p}_2$$

$$\bar{\mathcal{I}}_1 = (\mathcal{C}_1 + \mathcal{C}_2)/\tilde{\Gamma}_{G_1} \qquad \bar{\mathcal{I}}_2 = (\mathcal{C}_1 + \mathcal{C}_2)/\tilde{\Gamma}_{G_2}\ .$$

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx, dv_3 \wedge dx$$
$$dw - w_1 dy,\ dw_1 - w_2\,dy,\ dw_2 - w_3\,dy, dw_3 \wedge dx\rangle$$

$$X_1 = \partial_v - \partial_w,\ X_2 = pr(v\,\partial_v + w\,\partial_w),\ X_3 = pr(v^2\partial_v - w^2\partial_w),\ X_4 = \partial_v + \partial_w,$$

Let

$$\Gamma_{G_2} = \{X_1, X_2, X_3\}$$
$$\Gamma_{G_1} = \{X_1, X_2, X_4\}$$
$$\Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \{X_1, X_2\}$$

and construct the commutative diagram.

Note: $\mathcal{I}/\Gamma_{G_2}$ was the Pfaffian system for Liouville's equation $u_{xy} = e^u$.

Example Part H: The $\Gamma_H = \{X_1, X_2\}$ reduction

$$M = J^3(\mathbf{R}, \mathbf{R}) \times J^3(\mathbf{R}, \mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

$$\mathcal{I} = \langle dv - v_1\,dx,\; dv_1 - v_2\,dx,\; dv_2 - dv_3\,dx,\; dv_3 \wedge dx$$
$$dw - w_1\,dy,\; dw_1 - w_2\,dy,\; dw_2 - w_3\,dy,\; dw_3 \wedge dy\rangle$$

$$X_1 = \partial_v - \partial_w,\; X_2 = pr(v\,\partial_v + w\,\partial_w),\; X_3 = pr(v^2\partial_v - w^2\partial_w),\; X_4 = \partial_v + \partial_w,$$

$$\boxed{J^3(\mathbf{R}, \mathbf{R}) \times J^3(\mathbf{R}, \mathbf{R})} \xrightarrow{\{X_1, X_2\}} \boxed{V_y = -e^W, W_x = -e^V}$$

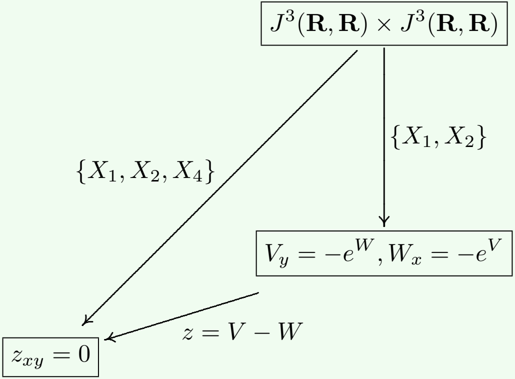
$\mathcal{I}/\Gamma_H$ - Rank 4 Pfaffian system on an 8 manifold, M/Γ_H

Example Part G_1: The $\Gamma_{G_1} = \{X_1, X_2, X_4\}$ reduction

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

$$\mathcal{I} = \langle dv - v_1\, dx,\ dv_1 - v_2\, dx,\ dv_2 - dv_3\, dx,$$
$$dw - w_1 dy,\ dw_1 - w_2\, dy,\ dw_2 - w_3\, dy\rangle$$

$$X_1 = \partial_v - \partial_w,\ X_2 = pr(v\,\partial_v + w\,\partial_w),\ X_3 = pr(v^2\partial_v - w^2\partial_w),\ X_4 = \partial_v + \partial_w,$$



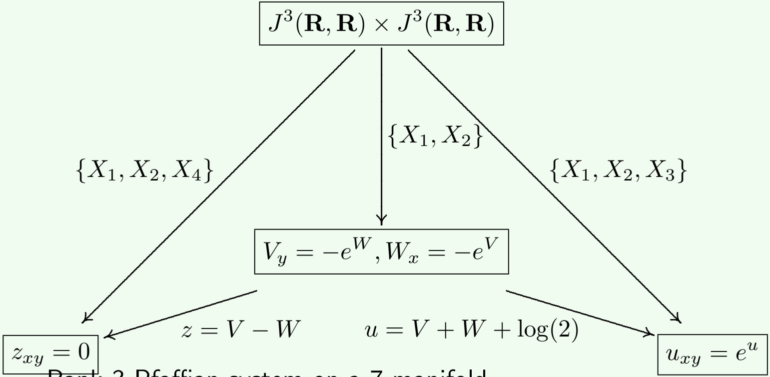
$\mathcal{I}/\Gamma_{G_1}$ - Rank 3 Pfaffian system on a 7-manifold, M/Γ_{G_1}

Example Part G_2: The $\Gamma_{G_2} = \{X_1, X_2, X_3\}$ reduction

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

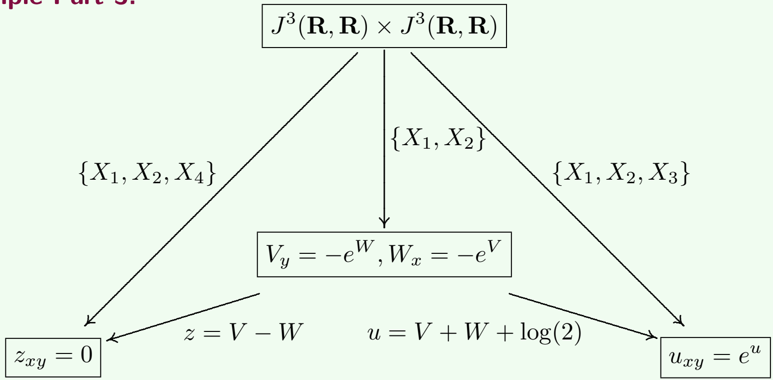
$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx,\ dv_3 \wedge dx$$
$$dw - w_1\,dy,\ dw_1 - w_2\,dy,\ dw_2 - w_3\,dy,\ dw_3 \wedge dy\rangle$$

$$X_1 = \partial_v - \partial_w,\ X_2 = pr(v\,\partial_v + w\,\partial_w),\ X_3 = pr(v^2\partial_v - w^2\partial_w),\ X_4 = \partial_v + \partial_w,$$



$\mathcal{I}/\Gamma_{G_2}$ - Rank 3 Pfaffian system on a 7-manifold.

Example Part 3:



$$z = V - W, \qquad z_x = V_x - W_x, \quad z_y = V_y - W_y$$
$$u = V + W + \log 2, \qquad u_x = V_x + W_x, \quad u_y = V_y + W_y$$

Writing $W_x = e^{-V}$ and $V_y = e^{-W}$ gives

$$z_x - u_x = \sqrt{2}\exp\frac{z+u}{2}, \quad z_y + u_y = -\sqrt{2}\exp\frac{u-z}{2}$$

Example Part 3: Deprolongation

$$M = J^3(\mathbf{R},\mathbf{R}) \times J^3(\mathbf{R},\mathbf{R}) = \mathbf{R}^{10}(x, v, v_1, v_2, v_3, y, w, w_1, w_2, w_3).$$

$$\mathcal{I} = \langle dv - v_1\,dx,\ dv_1 - v_2\,dx,\ dv_2 - dv_3\,dx,$$
$$dw - w_1 dy,\ dw_1 - w_2\,dy,\ dw_2 - w_3\,dy\rangle$$

$X_1 = \partial_v - \partial_w,\ X_2 = pr(v\,\partial_v + w\,\partial_w),\ X_3 = pr(v^2\partial_v - w^2\partial_w),\ X_4 = \partial_v + \partial_w,$

The quotients $\mathcal{I}/\Gamma_{G_a}$ are rank 3 Pfaffian systems on the seven manifolds

$$\Gamma_{G_1} = \{X_1, X_2, X_3\},\ \Gamma_{G_2} = \{X_1, X_2, X_4\},\ \Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \{X_1, X_2\}.$$

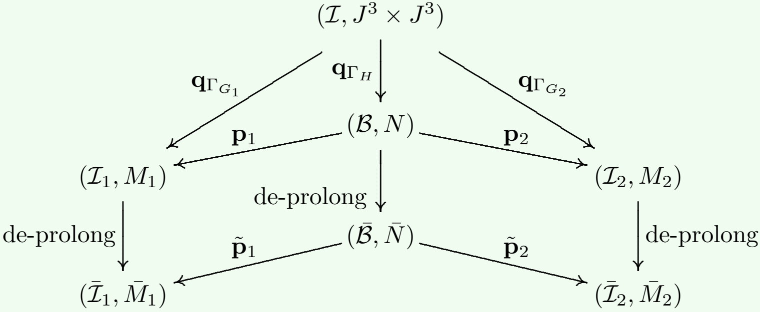
$\mathcal{I}/\Gamma_{G_a}$ are rank 3 Pfaffian systems on the 7 manifolds M/Γ_{G_a}

$\mathcal{I}/\Gamma_H$ is a rank 4 Pfaffian system on the 8 manifold M/Γ_H.

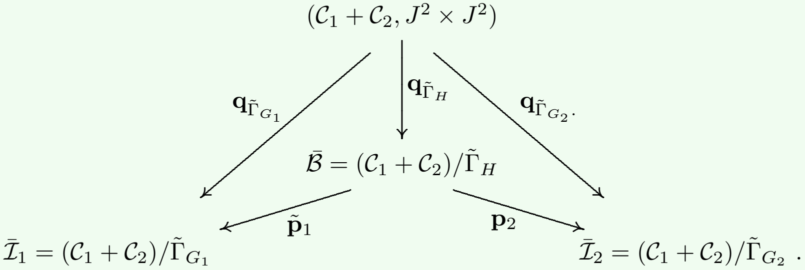
Our diagram can be de-prolonged.

Let $\bar{\mathcal{I}}_a$ be the Monge-Ampère representations on $z_{xy} = 0$ and $u_{xy} = e^u$.

Example Part 3:Deprolongation



de-prolongation of $(\mathcal{K}_1 + \mathcal{K}_2, J^3 \times J^3)$ gives,



Monge-Ampère Systems

Theorem - Let $\mathcal{B}$ be a (local) Bäcklund transformation, with one-dimensional fibers, between a hyperbolic Monge-Ampère system $\mathcal{I}_2$ and the Monge-Ampère form of the wave equation $\mathcal{I}_{z_{xy}} = 0$. Then Bäcklund transformation $\mathcal{B}$ can be constructed locally as a group quotient in accordance with Theorem A in essentially all but one case:

$$u_{xy} = \frac{\sqrt{1 - u_x^2}\,\sqrt{1 - u_y^2}}{\sin u}.$$

This equation is an $SO(3)$ quotient, and since $SO(3)$ has no two dimensional subgroups, no such Bäcklund transformation exists.