Moving Frame methods for solving $SE(3)$ symmetric variational problems

Tânia Gonçalves
Joint work with Elizabeth Mansfield

Moving Frames in Geometry

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University of Kent

Introduction

Noether's First Theorem yields conservation laws for Lagrangians with a variational symmetry group.

Recently, we proved that Noether's conservation laws can be written as the divergence of the product of a moving frame and a vector of invariants.

Interesting fact New format for Noether's conservation laws reduces the integration problem.

How do these conservation laws simplify one-dimensional variational problems which are invariant under the special Euclidean group $SE(3)$?

Outline

- Moving frames[1]
- Invariant calculus of variations[2]
- Noether's Theorem
- Solution of $SE(3)$ symmetric variational problems

[1] M. Fels and P.J. Olver, Acta Appl. Math. **51** (1998) and **55** (1999)
[2] I.A. Kogan and P.J. Olver, Acta Appl. Math. **76** (2003)

Here we will use the notion of Cartan's moving frame as reformulated by Fels and Olver.

Consider a group G acting on the n-th jet bundle $J^n(X \times U)$, whose action is free and regular.

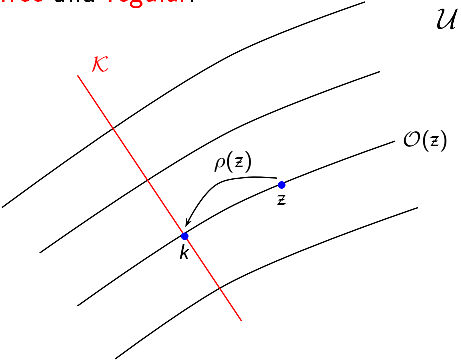


Figure: A local foliation with a transverse cross-section $\rho : \mathcal{U} \to G$

Moving frame calculation

The cross-section $\mathcal{K}$ is the locus of $\mathbf{\Psi}(z) = 0$. To obtain the frame $\rho(z)$ we solve the system

$$\Psi_j(g \cdot z) = 0, \qquad j = 1, ..., r = \dim(G)$$

for the r independent parameters describing G, in other words we solve the normalisation equations. By the IFT, a unique solution of $\mathbf{\Psi}(g \cdot z) = 0$ yields

$$\rho(g \cdot z) = \rho(z) g^{-1}, \qquad \textit{or} \qquad \rho(g \cdot z) = g^{-1} \rho(z),$$

i.e. $\rho(z)$ is equivariant.

Example Consider $SL(2)$ acting on $(x, t, u(x, t))$ as follows

$$g \cdot x = x, \quad g \cdot t = t, \quad g \cdot u = \frac{au + b}{cu + d},$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

The induced action on u_x, and similarly for other derivatives of u, is defined to be

$$g \cdot u_x = \frac{\partial(g \cdot u)}{\partial(g \cdot x)} = \frac{u_x}{(cu + d)^2}$$

by the chain rule.

Let $z = (u, u_x, u_{xx})$ and take $\mathbf{\Psi}(g \cdot z) = 0$ to be

$$g \cdot u = 0, \quad g \cdot u_x = 1, \quad g \cdot u_{xx} = 0.$$

Solving

$$a = \frac{1}{\sqrt{u_x}}, \quad b = -\frac{u}{\sqrt{u_x}}, \quad c = \frac{u_{xx}}{2u_x^{3/2}}.$$

Invariants The components of the cross-section $I(z) = \rho(z) \cdot z$ are invariant.

In our running example

$$I^u_{111} = g \cdot u_{xxx}\big|_{frame} = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}, \quad I^u_2 = g \cdot u_t\big|_{frame} = \frac{u_t}{u_x}$$

are the lowest order invariants. Let $\sigma = I^u_{111}$.

Various notations exist for the invariants in the literature

$$g \cdot u_K^\alpha|_{frame} = I_K^\alpha = \iota(u_K^\alpha) = \bar{\iota} u_K^\alpha.$$

Analogously, we have invariant differential operators

$$\mathcal{D}_j = \left.\frac{D}{D(g \cdot x_j)}\right|_{frame}.$$

In our running example

$$\mathcal{D}_x = \frac{D}{Dx}, \quad \mathcal{D}_t = \frac{D}{Dt},$$

and

$$[\mathcal{D}_x, \mathcal{D}_t] = 0.$$

All differential invariants are functions of the I^α_K by the Fels-Olver-Thomas Replacement Theorem:

If $f(z)$ is invariant, then

$$f(z) = f(g \cdot z) = f(\rho(z) \cdot z) = f(I(z)).$$

We know that

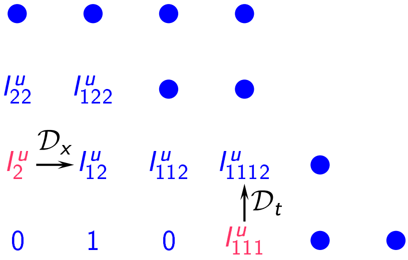
$$\frac{\partial}{\partial x_j} u^\alpha_K = u^\alpha_{Kj},$$

but $\mathcal{D}_j I^\alpha_K \neq I^\alpha_{Kj}$; indeed

$$\mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj},$$

where M^α_{Kj} are the error terms.

Example (cont.) We have two generators $\sigma = I^u_{111}$ and I^u_2, and to obtain I^u_{1112} we can use one of two paths.



Syzygy between the two generators

$$\mathcal{D}_t\sigma = (\mathcal{D}_x^3 + 2\sigma\mathcal{D}_x + \sigma_x)I^u_2.$$

Invariant calculus of variations

Recall, we want to use the invariantised versions of the Euler-Lagrange equations and Noether's conservation laws to find a solution for symmetric variational problems.

Recall how we calculate the Euler-Lagrange equations for one-dimensional Lagrangians

$$\begin{aligned}
0 &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathscr{L}[u+\varepsilon v] \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_a^b L(x, u+\varepsilon v, u_x+\varepsilon v_x, u_{xx}+\varepsilon v_{xx}, \dots)\,\mathrm{d}x \\
&= \int_a^b \left(\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_x} v_x + \frac{\partial L}{\partial u_{xx}} v_{xx} + \dots \right) \mathrm{d}x \\
&= \int_a^b \Bigg[\left(\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u_{xx}} + \cdots \right) v \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial u_x} v + \frac{\partial L}{\partial u_{xx}} v_x - \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_{xx}} \right) v + \dots \right) \Bigg] \mathrm{d}x \\
&= \int_b^a E(L) v \,\mathrm{d}x + \left[\frac{\partial L}{\partial u_x} v + \dots \right]_a^b
\end{aligned}$$

To get the invariantised Euler-Lagrange equations and Noether's conservation laws, we introduce a dummy invariant variable t and set $u = u(x, t)$. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathscr{L}[u^\alpha + \varepsilon v^\alpha] = \left.\frac{D}{Dt}\right|_{u_t^\alpha = v^\alpha} \mathscr{L}[u^\alpha]$$

yield the same symbolic result.

Example (cont.) Consider the one-dimensional Lagrangian with finite number of arguments

$$\mathscr{L}[u] = \int L(\sigma, \sigma_x, \sigma_{xx}, ...)\mathrm{d}x.$$

Introduce the dummy variable t to effect the variation. This gives a new invariant $I_2^u = u_t/u_x$ with syzygy

$$\mathcal{D}_t\sigma = (\mathcal{D}_x^3 + 2\sigma\mathcal{D}_x + \sigma_x)I_2^u = \mathcal{H}I_2^u.$$

Hence,

$$
\begin{aligned}
\mathcal{D}_t \int L(\sigma, \sigma_x, \sigma_{xx}, \dots)\,\mathrm{d}x
&= \int \left(\frac{\partial L}{\partial \sigma} + \frac{\partial L}{\partial \sigma_x}\mathcal{D}_x + \cdots \right) \mathcal{D}_t \sigma \,\mathrm{d}x \\
&= \int \underbrace{\left(\frac{\partial L}{\partial \sigma} - \mathcal{D}_x \frac{\partial L}{\partial \sigma_x} + \mathcal{D}_x^2 \frac{\partial L}{\partial \sigma_{xx}} + \cdots \right)}_{E^\sigma(L)} \mathcal{H}(I_2^u)\,\mathrm{d}x \\
&\quad + \left[\frac{\partial L}{\partial \sigma_x}\mathcal{D}_t\sigma + \frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_x\mathcal{D}_t\sigma - \mathcal{D}_x \frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_t\sigma + \cdots \right]_a^b \\
&= \int \mathcal{H}^* \left(E^\sigma(L) \right) I_2^u \,\mathrm{d}x \\
&\quad + \Big[E^\sigma(L)\mathcal{D}_x^2 I_2^u - \mathcal{D}_x E^\sigma(L)\mathcal{D}_x I_2^u + \mathcal{D}_x^2 E^\sigma(L) I_2^u + 2\sigma E^\sigma(L) I_2^u \\
&\quad + \frac{\partial L}{\partial \sigma_x}\mathcal{D}_t\sigma + \frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_x\mathcal{D}_t\sigma - \mathcal{D}_x \frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_t\sigma + \cdots \Big]_a^b ,
\end{aligned}
$$

where $\mathcal{H}^*$ is the adjoint of $\mathcal{H}$. So $E^u(L) = \mathcal{H}^* E^\sigma(L) = 0$.

Noether's First Theorem

Noether's Theorem provides first integrals of the Euler-Lagrange equations for one-dimensional variational problems that are invariant under a Lie group.

As shown before, we obtain Noether's conservation laws by carefully keeping track of the boundary terms.

Example (cont.) The conservation laws associated to $\int L(\sigma, \sigma_x, \sigma_{xx}, \ldots)\mathrm{d}x$ are

$$\underbrace{\begin{pmatrix} ad+bc & -2ab & 2cd \\ -ac & a^2 & -c^2 \\ bd & -b^2 & d^2 \end{pmatrix}}_{R(g)^{-1}}\Bigg|_{frame} \begin{pmatrix} -2\mathcal{D}_x E^\sigma(L) \\ \sigma E^\sigma(L) + \mathcal{D}_x^2 E^\sigma(L) \\ -2E^\sigma(L) \end{pmatrix} = \mathbf{c}.$$

Recall the frame is

$$a = \frac{1}{\sqrt{u_x}}, \quad b = -\frac{u}{\sqrt{u_x}}, \quad c = \frac{u_{xx}}{2u_x^{3/2}}, \quad ad - bc = 1.$$

$R(gh) = R(g)R(h)$, so $R(\rho(z))$ is equivariant.

Which representation yields $R(g)$? How do we calculate the vector of invariants?

Adjoint representation of $SL(2)$ with respect to the infinitesimal vector fields

For

$$g \cdot u = \frac{au+b}{cu+d}, \qquad where\ ad - bc = 1,$$

the infinitesimal vector fields are

$$2\partial_u, \quad \partial_u, \quad -u^2\partial_u.$$

Let $g \in SL(2)$ act on

$$(2\alpha u + \beta - \gamma u^2)\partial_u,$$

where α, β and γ are constants.

Thus,

$$
\begin{aligned}
& g \cdot (2\alpha u + \beta - \gamma u^2)\partial_u \\
= \;& (2\alpha(g \cdot u) + \beta - \gamma(g \cdot u)^2)\partial_{(g\cdot u)} \\
= \;& \left(2\alpha\tfrac{au+b}{cu+d} + \beta - \gamma\left(\left(\tfrac{au+b}{cu+d}\right)^2\right)\right)(cu+d)^2\partial_u \\
= \;& \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} ad+bc & 2bd & -2ac \\ cd & d^2 & -c^2 \\ -ab & -b^2 & a^2 \end{pmatrix}}_{R(g)} \begin{pmatrix} 2u\partial_u \\ \partial_u \\ -u^2\partial_u \end{pmatrix}.
\end{aligned}
$$

Recall the collection of boundary terms

$$E^{\sigma}(L)\mathcal{D}_x^2 I_2^u - \mathcal{D}_x E^{\sigma}(L)\mathcal{D}_x I_2^u + \mathcal{D}_x^2 E^{\sigma}(L) I_2^u + 2\sigma E^{\sigma}(L) I_2^u$$

$$+\frac{\partial L}{\partial \sigma_x}\mathcal{D}_t\sigma + \frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_x\mathcal{D}_t\sigma - \mathcal{D}_x\frac{\partial L}{\partial \sigma_{xx}}\mathcal{D}_t\sigma + \cdots = k,$$

where k is a constant. Substituting $\mathcal{D}_x^2 I_2^u$, $\mathcal{D}_x I_2^u$ etc. in the above by their differential formulae,

$$\mathcal{D}_x^2 I_2^u = I_{112}^u - \sigma I_2^u,$$

$$\mathcal{D}_x I_2^u = I_{12}^u,$$

$$\mathcal{D}_t\sigma = I_{1112}^u - \sigma I_{12}^u,$$

$$\mathcal{D}_x\mathcal{D}_t\sigma = I_{11112}^u - 4\sigma I_{112}^u - \sigma_x I_{12}^u,$$

$$\vdots$$

we obtain the conservation law in the form, linear in the I^u_{2J},

$$\begin{pmatrix} I^u_2 & I^u_{12} & \cdots \end{pmatrix} \underbrace{\begin{pmatrix} \mathcal{D}_x^2 E^\sigma(L) + \sigma E^\sigma(L) \\ -\mathcal{D}_x E^\sigma(L) + \cdots \\ E^\sigma(L) + \cdots \\ \vdots \end{pmatrix}}_{\mathcal{C}^u} = k.$$

Multiplying the vector $\mathcal{C}^u$ by the matrix of invariantised infinitesimals, $\Omega^u(I)$, we obtain the vector of invariants

$$\upsilon(I) = \begin{pmatrix} -2\mathcal{D}_x E^\sigma(L) \\ \sigma E^\sigma(L) + \mathcal{D}_x^2 E^\sigma(L) \\ -2E^\sigma(L) \end{pmatrix}.$$

Noether's Theorem

Theorem Let $\int L(\kappa_1, \kappa_2, ...)\mathrm{d}s$ be invariant under $G \times M \to M$ with generating invariants κ_j, for $j = 1, ..., N$, and let $\widetilde{x_i} = x_i$, for $i = 1, ..., p$. Introduce a dummy variable t to effect the variation and suppose that

$$\mathcal{D}_t \int L(\kappa_1, \kappa_2, ...)\mathrm{d}\mathbf{x} = \int \Big[\sum_{j,\alpha} \mathcal{H}^*_{j,\alpha} \mathrm{E}^j(L) I^\alpha_t + \mathrm{Div}(P) \Big] \mathrm{d}\mathbf{x}, \quad (1)$$

where this defines a p-tuple P, whose components are of the form

$$P_i = \sum_{\alpha, J} I^\alpha_{tJ} C^\alpha_{J,i} = \sum_{\alpha, m, J} \mathcal{A}d(\rho)^{-1}_{km} \Omega^\alpha(I)_{mJ} C^\alpha_{J,i}, \qquad i = 1, ..., p$$

and the vectors $\mathcal{C}^\alpha_i = (C^\alpha_{J,i})$. Hence the the r conservation laws obtained via Noether's First Theorem can be written in the form

$$\sum_i \mathcal{D}_{x_i} \left(\mathcal{A}d^{-1}_\rho \boldsymbol{\upsilon}_i(I) \right) = 0,$$

where $\mathcal{A}d^{-1}_\rho$ is $\mathcal{A}d(g)$ evaluated at the frame and $\boldsymbol{\upsilon}(I) = \sum_\alpha \Omega^\alpha(I) \mathcal{C}^\alpha_i$.

Solution of $SE(3)$ symmetric variational problems

Consider the $SE(3)$ group action on the $(x(s), y(s), z(s))$-space, parametrised by the Euclidean arc length, given by

$$\widetilde{\mathbf{x}(s)} = \mathrm{R}^{-1}(\mathbf{x}(s) - \mathbf{a}),$$

where $\mathbf{x}(s) = (x(s), y(s), z(s))^T$, R^{-1} is a three-dimensional rotation, and $\mathbf{a} = (a, b, c)$ a translation vector.

The normalisation equations that define the moving frame are

$$\widetilde{x} = 0, \; \widetilde{y} = 0, \; \widetilde{z} = 0, \; \widetilde{y_s} = 0, \; \widetilde{z_s} = 0, \text{ and } \widetilde{z_{ss}} = 0.$$

Solving these normalisation equations gives us the frame in parametric form

$$a = x,\ b = y,\ c = z,\ \theta = \arctan\left(\frac{y_s}{x_s}\right),\ \nu = \arctan\left(\frac{z_s}{\sqrt{x_s^2 + y_s^2}}\right),$$

$$\alpha = \arctan\left(\frac{z_{ss}}{(x_s y_{ss} - y_s x_{ss})\sqrt{x_s^2 + y_s^2 + z_s^2}}\right).$$

The infinitesimal vector fields generating $SE(3)$ are

$$\mathbf{f}_1 \partial_x,\ \mathbf{f}_2 = \partial_y,\ \mathbf{f}_3 = \partial_z,\ \mathbf{f}_4 = y\partial_z - z\partial_y,$$

$$\mathbf{f}_5 = x\partial_z - z\partial_x,\ \mathbf{f}_6 = x\partial_y - y\partial_x.$$

Letting $g \in SE(3)$ act on
$\mathbf{f} = p_1\mathbf{f}_1 + p_2\mathbf{f}_2 + p_3\mathbf{f}_3 + p_4\mathbf{f}_4 + p_5\mathbf{f}_5 + p_6\mathbf{f}_6$ gives us the representation for g, $\mathcal{A}d(g)$. Evaluating $\mathcal{A}d(g)$ at the frame provides

$$\mathcal{A}d_\rho^{-1} = \begin{pmatrix} \mathcal{R}^T & O_{3\times 3} \\ D\,T\mathcal{R}^T & D\mathcal{R}^T D \end{pmatrix},$$

where

$$\mathcal{R}^T = \begin{pmatrix} x_s & \dfrac{x_{ss}}{\kappa} & \dfrac{k_1}{\kappa} \\ y_s & \dfrac{y_{ss}}{\kappa} & \dfrac{k_2}{\kappa} \\ z_s & \dfrac{z_{ss}}{\kappa} & \dfrac{k_3}{\kappa} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

and $k_1 = y_s z_{ss} - z_s y_{ss}$, $k_2 = z_s x_{ss} - x_s z_{ss}$, $k_3 = x_s y_{ss} - y_s x_{ss}$, $D = diag(1, -1, 1)$, and $O_{3\times 3}$ is the zero matrix.

Let $\mathscr{L}[\kappa,\tau]=\int[L(\kappa,\kappa_s,\tau,\tau_s)-\lambda(s)(\eta-1)]\,\mathrm{d}s$ be the invariantised Lagrangian under the $SE(3)$ group action, where κ is the Euclidean curvature, τ is the torsion and $\eta=\sqrt{x_s^2+y_s^2+z_s^2}$. Differentiation and integration by parts of the invariantised Lagrangian yields two invariantised Euler-Lagrange equations in two unknowns, after using $\mathrm{E}^x(L)=0$ to eliminate λ

$$\begin{aligned}\mathrm{E}^y(L)=&(\kappa^2-\tau^2)\mathrm{E}^\kappa(L)+2\tau\kappa\mathrm{E}^\tau(L)-2L\kappa+\kappa\kappa_s\frac{\partial L}{\partial\kappa_s}+\kappa\tau_s\frac{\partial L}{\partial\tau_s}\\&+\left(\frac{\tau_s}{\kappa}-\frac{2\tau\kappa_s}{\kappa^2}\right)\mathcal{D}_s\mathrm{E}^\tau(L)+\mathcal{D}_s^2\mathrm{E}^\kappa(L)+\frac{2\tau}{\kappa}\mathcal{D}_s^2\mathrm{E}^\tau(L)=0,\\\mathrm{E}^z(L)=&-\kappa_s\mathrm{E}^\tau(L)+\left(-\kappa+\frac{\tau^2}{\kappa}-\frac{2\kappa_s^2}{\kappa^3}+\frac{\kappa_{ss}}{\kappa^2}\right)\mathcal{D}_s\mathrm{E}^\tau(L)\\&+\frac{2\kappa_s}{\kappa^2}\mathcal{D}_s^2\mathrm{E}^\tau(L)-\frac{1}{\kappa}\mathcal{D}_s^3\mathrm{E}^\tau(L)+\tau_s\mathrm{E}^\kappa(L)+2\tau\mathcal{D}_s\mathrm{E}^\kappa(L)=0,\end{aligned}$$

and the coefficients of I^α_{tJ} in the boundary terms, $\mathcal{C}^\alpha$.

To obtain the invariantised Euler-Lagrange equations and the p-tuple P, we had to use the following syzygies

$$\mathcal{D}_t\eta = \mathcal{D}_s I_2^x - \kappa I_2^y,$$

$$\mathcal{D}_t\kappa = \mathcal{D}_s^2 I_2^y - 2\tau\mathcal{D}_s I_2^z + \kappa_s I_2^x + (\kappa^2 - \tau^2) I_2^y - \tau_s I_2^z,$$

$$\begin{aligned}\mathcal{D}_t\tau = {} & \frac{1}{\kappa}\mathcal{D}_s^3 I_2^z + \frac{2\tau}{\kappa}\mathcal{D}_s^2 I_2^y - \frac{\kappa_s}{\kappa^2}\mathcal{D}_s^2 I_2^z + 2\tau\mathcal{D}_s I_2^x + \left(\frac{3\tau_s}{\kappa} - \frac{2\kappa_s\tau}{\kappa^2}\right)\mathcal{D}_s I_2^y \\ & + \left(\kappa - \frac{\tau^2}{\kappa}\right)\mathcal{D}_s I_2^z + \tau_s I_2^x + \left(\frac{\tau_{ss}}{\kappa} - \frac{\tau_s\kappa_s}{\kappa^2}\right) I_2^y + \left(\frac{\kappa_s\tau^2}{\kappa^2} - \frac{2\tau\tau_s}{\kappa}\right) I_2^z,\end{aligned}$$

and the differential formulae

$$\mathcal{D}_t\kappa = -2\kappa I_{12}^x = I_{112}^y,$$

$$\mathcal{D}_t\tau = \tau I_{12}^x - \frac{\tau}{\kappa} I_{112}^y + \kappa I_{12}^z - \frac{\kappa_s}{\kappa^2} I_{112}^z + \frac{1}{\kappa} I_{1112}^z,$$

$$\mathcal{D}_s I_2^y = -\kappa I_2^x + I_{12}^y + \tau I_2^z,$$

$$\mathcal{D}_s I_2^z = -\tau I_2^y + I_{12}^z,$$

$$\mathcal{D}_s^2 I_2^z = \tau\kappa I_2^x - \tau_s I_2^y - 2\tau I_{12}^y - \tau^2 I_2^z + I_{112}^z.$$

Solution of $SE(3)$ symmetric variational problems

Multiplying the vectors $\mathcal{C}^\alpha$ for $\alpha = 1, ..., 3$, by the respective matrices of invariantised infinitesimals, $\Omega^\alpha(I)$ and then adding them up gives us the vector of invariants

$$\boldsymbol{\upsilon}(I) = \begin{pmatrix} -\kappa \mathrm{E}^\kappa(L) - \tau \mathrm{E}^\tau(L) + 2L - \kappa_s \frac{\partial L}{\partial \kappa_s} - \tau_s \frac{\partial L}{\partial \tau_s} \\ -\mathcal{D}_s \mathrm{E}^\kappa(L) - \frac{\tau}{\kappa}\mathcal{D}_s \mathrm{E}^\tau(L) \\ \kappa \mathrm{E}^\tau(L) + \frac{1}{\kappa}\mathcal{D}_s^2 \mathrm{E}^\tau(L) - \tau \mathrm{E}^\kappa(L) - \frac{\kappa_s}{\kappa^2}\mathcal{D}_s \mathrm{E}^\tau(L) \\ \mathrm{E}^\tau(L) \\ -\frac{1}{\kappa}\mathcal{D}_s \mathrm{E}^\tau(L) \\ \mathrm{E}^\kappa(L) \end{pmatrix}$$

The conservation laws are $\mathcal{A}d_\rho^{-1}\boldsymbol{\upsilon}(I) = \mathbf{c}$, where $\mathbf{c} = (\mathbf{c_1}, \mathbf{c_2})^T$ is the constant vector with $\mathbf{c_1} = (c_1, c_2, c_3)^T$ and $\mathbf{c_2} = (c_4, c_5, c_6)^T$.

Using the conservation laws $\mathcal{A}d(\rho)^{-1}\boldsymbol{\upsilon}(I) = \mathbf{c}$ we get a first integral of the Euler-Lagrange equations

$$\left(-\tau \mathrm{E}^{\tau}(L) - \kappa \mathrm{E}^{\kappa}(L) + 2L - \frac{\partial L}{\partial \kappa_s}\kappa_s - \frac{\partial L}{\partial \tau_s}\tau_s\right)^2 + \left(-\mathcal{D}_s \mathrm{E}^{\kappa}(L) - \frac{\tau}{\kappa}\mathcal{D}_s \mathrm{E}^{\tau}(L)\right)^2$$
$$+\left(\frac{1}{\kappa}\mathcal{D}_s^2 \mathrm{E}^{\tau}(L) - \frac{\kappa_s}{\kappa^2}\mathcal{D}_s \mathrm{E}^{\tau}(L) + \kappa \mathrm{E}^{\tau}(L) - \tau \mathrm{E}^{\kappa}(L)\right)^2 = c_1^2 + c_2^3 + c_3^2.$$

How are these conservation laws going to help reduce the integration problem?

First we simplify the conservation laws in two steps.

First step
Apply an element of $SE(3)$, say $\mathcal{A}d(g)^{-1}$, to both sides of $\mathcal{A}d_\rho(z)^{-1}\boldsymbol{\upsilon}(I) = \mathbf{c}$ such that it sends $\mathbf{c_1}$ and $\mathbf{c_2}$ to the z-axis.

But how does $\mathcal{A}d(g)$ act on the vector $\mathbf{c}$?

$$\mathcal{A}d(g)\mathbf{c} = \begin{pmatrix} R & \mathbf{0} \\ DTR & DRD \end{pmatrix} \begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \end{pmatrix}$$

The Adjoint representation of G does not act freely on the constant vector $\mathbf{c}$, since it preserves the length of $\mathbf{c_1}$ and the quantity ${\mathbf{c_1}}^T D\mathbf{c_2}$, as shown below

$$DTR\mathbf{c_1} + DRD\mathbf{c_2} = \widetilde{\mathbf{c_2}}$$

$$T\widetilde{\mathbf{c_1}} + RD\mathbf{c_2} = D\widetilde{\mathbf{c_2}}$$

$${\mathbf{c_1}}^T R^T T\widetilde{\mathbf{c_1}} + {\mathbf{c_1}}^T D\mathbf{c_2} = {\mathbf{c_1}}^T R^T D\widetilde{\mathbf{c_2}}$$

$$\underbrace{\widetilde{\mathbf{c_1}}^T T\widetilde{\mathbf{c_1}}}_{=0} + {\mathbf{c_1}}^T D\mathbf{c_2} = \widetilde{\mathbf{c_1}}^T D\widetilde{\mathbf{c_2}}$$

So let $\mathcal{A}d(g)^{-1}$ act on $\mathbf{c}$ to obtain

$$\mathbf{C} = \left(0, 0, |\mathbf{c_1}|, 0, 0, \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|}\right)^T,$$

generic case, where $\mathbf{c_1} \neq 0$. Hence,

$$\mathcal{A}d_\rho(\widetilde{z})^{-1}\boldsymbol{v}(I) = \mathbf{C}, \tag{2}$$

by the equivariance of the right moving frame, i.e.

$$\mathcal{A}d_\rho(g \cdot z)^{-1} = \mathcal{A}d(g)^{-1}\mathcal{A}d_\rho(z)^{-1}.$$

Second step

Next, applying $\mathcal{A}d_\rho(\widetilde{z})$ to both sides of (2) gives us

$$\mathcal{A}d_\rho(\widetilde{z})\mathbf{C} = \boldsymbol{\upsilon}(I),$$

more precisely

$$|\mathbf{c}_1|\widetilde{z_s} = \upsilon^{(1)}(I), \tag{3}$$

$$\frac{|\mathbf{c}_1|}{\kappa}\widetilde{z_{ss}} = \upsilon^{(2)}(I), \tag{4}$$

$$\frac{|\mathbf{c}_1|}{\kappa}(\widetilde{x_s}\widetilde{y_{ss}} - \widetilde{y_s}\widetilde{x_{ss}}) = \upsilon^{(3)}(I), \tag{5}$$

$$|\mathbf{c}_1|(\widetilde{x}\widetilde{y_s} - \widetilde{y}\widetilde{x_s}) + \frac{\mathbf{c}_1{}^T D\mathbf{c}_2}{|\mathbf{c}_1|}\widetilde{z_s} = \upsilon^{(4)}(I), \tag{6}$$

$$\frac{|\mathbf{c}_1|}{\kappa}(\widetilde{x_{ss}}\widetilde{y} - \widetilde{y_{ss}}\widetilde{x}) - \frac{\mathbf{c}_1{}^T D\mathbf{c}_2}{\kappa|\mathbf{c}_1|}\widetilde{z_{ss}} = \upsilon^{(5)}(I), \tag{7}$$

$$\frac{|\mathbf{c}_1|}{\kappa}(\widetilde{x}(\widetilde{z_s}\widetilde{x_{ss}} - \widetilde{x_s}\widetilde{z_{ss}}) - \widetilde{y}(\widetilde{y_s}\widetilde{z_{ss}} - \widetilde{z_s}\widetilde{y_{ss}})) + \frac{\mathbf{c}_1{}^T D\mathbf{c}_2}{\kappa|\mathbf{c}_1|}(\widetilde{x_s}\widetilde{y_{ss}} - \widetilde{y_s}\widetilde{x_{ss}}) = \upsilon^{(6)}(I), \tag{8}$$

where we have used $\upsilon^{(j)}(I)$ to denote the j-th component of $\boldsymbol{\upsilon}(I)$.

Once we have solved for κ and τ, we can solve this overdetermined system for the original variables.

So starting with Equation (3),

$$\widetilde{z_s} = \frac{1}{|\mathbf{c_1}|} \upsilon^{(1)}(I),$$

we obtain that

$$\widetilde{z(s)} = \frac{1}{|\mathbf{c_1}|} \int \upsilon^{(1)}(I) \mathrm{d}s.$$

Next, multiplying Equation (5) by $-\frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}$ and adding it to Equation (8) gives

$$\frac{|\mathbf{c_1}|}{\kappa}(\widetilde{x}(\widetilde{z_s}\widetilde{x_{ss}} - \widetilde{x_s}\widetilde{z_{ss}}) - \widetilde{y}(\widetilde{y_s}\widetilde{z_{ss}} - \widetilde{z_s}\widetilde{y_{ss}})) = \upsilon^{(6)}(I) - \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}\upsilon^{(3)}(I),$$

which simplifies to

$$|\mathbf{c_1}|\left(\widetilde{z_s}\left(\frac{1}{2}\mathcal{D}_s^2(\widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x}}) - 1\right) - \frac{1}{2}\widetilde{z_{ss}}\mathcal{D}_s(\widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x}})\right) = \kappa\upsilon^{(6)}(I) - \kappa\frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}\upsilon^{(3)}(I),$$

where $\frac{1}{2}\mathcal{D}_s^2(\widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x}}) - 1 = \widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x_{ss}}}$ and $\frac{1}{2}\mathcal{D}_s(\widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x}}) = \widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x_s}}$. Now let $\mathcal{D}_s(\widetilde{\mathbf{x}}\cdot\widetilde{\mathbf{x}}) = h(s)$ and substitute $\widetilde{z_s}$ and $\widetilde{z_{ss}}$ respectively by $\frac{1}{|\mathbf{c_1}|}\upsilon^{(1)}(I)$ and $\frac{1}{|\mathbf{c_1}|}\mathcal{D}_s\upsilon^{(1)}(I)$. Rearranging we obtain the following equation linear for h

$$\mathcal{D}_s h - \frac{\mathcal{D}_s\upsilon^{(1)}(I)}{\upsilon^{(1)}(I)}h = 2\kappa\left(\upsilon^{(6)}(I) - \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}\upsilon^{(3)}(I)\right)\Big/\upsilon^{(1)}(I) + 2.$$

Solution of $SE(3)$ symmetric variational problems

Solving for h gives us

$$h(s) = \upsilon^{(1)}(I) \int \frac{1}{\upsilon^{(1)}(I)} \left(2\kappa \left(\upsilon^{(6)}(I) - \frac{\mathbf{c_1}^T D \mathbf{c_2}}{|\mathbf{c_1}|^2} \upsilon^{(3)}(I) \right) \Big/ \upsilon^{(1)}(I) + 2 \right) \mathrm{d}s.$$

Now we know that

$$\mathcal{D}_s(\widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{x}}) = \upsilon^{(1)}(I) \int \frac{1}{\upsilon^{(1)}(I)} \left(2\kappa \left(\upsilon^{(6)}(I) - \frac{\mathbf{c_1}^T D \mathbf{c_2}}{|\mathbf{c_1}|^2} \upsilon^{(3)}(I) \right) \Big/ \upsilon^{(1)}(I) + 2 \right) \mathrm{d}s. \tag{9}$$

Using cylindrical coordinates

$$\widetilde{x(s)} = r(s) \cos\theta(s), \qquad \widetilde{y(s)} = r(s) \sin\theta(s), \qquad \widetilde{z(s)} = \widetilde{z(s)},$$

Equation (9) gives us that

$$r(s)^2 = \int \left[\upsilon^{(1)}(I) \int \frac{1}{\upsilon^{(1)}(I)} \left(2\kappa \left(\upsilon^{(6)}(I) - \frac{\mathbf{c_1}^T D \mathbf{c_2}}{|\mathbf{c_1}|^2} \upsilon^{(3)}(I) \right) \Big/ \upsilon^{(1)}(I) + 2 \right) \mathrm{d}s \right] \mathrm{d}s - \left(\int \frac{\upsilon^{(1)}(I)}{|\mathbf{c_1}|} \mathrm{d}s \right)^2.$$

Finally using cylindrical coordinates to simplify Equation (6)

$$|\mathbf{c_1}|(\widetilde{x}\widetilde{y}_s - \widetilde{y}\widetilde{x}_s) + \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|}\widetilde{z}_s = \upsilon^{(4)}(I)$$

we obtain

$$r(s)^2\theta_s = \frac{1}{|\mathbf{c_1}|}\left(\upsilon^{(4)}(I) - \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}\upsilon^{(1)}(I)\right).$$

Hence,

$$\theta(s) = \int \frac{1}{r(s)^2|\mathbf{c_1}|}\left(\upsilon^{(4)}(I) - \frac{\mathbf{c_1}^T D\mathbf{c_2}}{|\mathbf{c_1}|^2}\upsilon^{(1)}(I)\right) \mathrm{d}s.$$

To recover x, y and z, we act on $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{z}$ as follows

$$\widetilde{\mathbf{x}} \mapsto \mathbf{x} = \mathrm{R}\widetilde{\mathbf{x}} + \mathbf{a}, \tag{10}$$

where R is a three-dimensional rotation and $\mathbf{a}$ is the translation vector with

$$\alpha = -\tan^{-1}\left(\frac{\sqrt{|\mathbf{c}_1|^2\cos^2\beta - c_3^2}}{c_3}\right),$$

$$\gamma = \tan^{-1}\left(\frac{c_2 c_3 \sin\beta + c_1\sqrt{|\mathbf{c}_1|^2\cos^2\beta - c_3^2}}{c_1 c_3 \sin\beta - c_2\sqrt{|\mathbf{c}_1|^2\cos^2\beta - c_3^2}}\right),$$

$$a = \frac{c_1}{c_3}c + \frac{c_5|\mathbf{c}_1|^2 + c_2\mathbf{c}_1{}^T D\mathbf{c}_2}{c_3|\mathbf{c}_1|^2}, \quad b = \frac{c_2}{c_3}c + \frac{c_4|\mathbf{c}_1|^2 - c_1\mathbf{c}_1{}^T D\mathbf{c}_2}{c_3|\mathbf{c}_1|^2},$$

and where β and c are free.

Although only four equations of the system have been used to solve x, y and z, we know that the remaining equations have been satisfied.

If we differentiate $\mathcal{A}d(\rho(z))^{-1}\boldsymbol{\upsilon}(I) = \mathbf{c}$ with respect to s and rearrange we obtain

$$\mathcal{D}_s\boldsymbol{\upsilon}(I) = \mathcal{D}_s\mathcal{A}d(\rho(z))\mathcal{A}d(\rho(z))^{-1}\boldsymbol{\upsilon}(I),$$

which is equivalent to

$$\mathcal{D}_s \begin{pmatrix} \upsilon^{(1)}(I) \\ \upsilon^{(2)}(I) \\ \upsilon^{(3)}(I) \\ \upsilon^{(4)}(I) \\ \upsilon^{(5)}(I) \\ \upsilon^{(6)}(I) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 & 0 & 0 & 0 \\ -\kappa & 0 & \tau & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\kappa & 0 \\ 0 & 0 & -1 & \kappa & 0 & -\tau \\ 0 & -1 & 0 & 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \upsilon^{(1)}(I) \\ \upsilon^{(2)}(I) \\ \upsilon^{(3)}(I) \\ \upsilon^{(4)}(I) \\ \upsilon^{(5)}(I) \\ \upsilon^{(6)}(I) \end{pmatrix}. \quad (11)$$

Solution of $SE(3)$ symmetric variational problems

The system (11) is part of an elimination ideal as it only involves invariants.

Hence, Equations (4),

$$\frac{|\mathbf{c_1}|}{\kappa}\widetilde{z_{ss}} = \upsilon^{(2)}(I),$$

and (7),

$$\frac{|\mathbf{c_1}|}{\kappa}(\widetilde{x_{ss}}\widetilde{y} - \widetilde{y_{ss}}\widetilde{x}) - \frac{\mathbf{c_1}^T D\mathbf{c_2}}{\kappa|\mathbf{c_1}|}\widetilde{z_{ss}} = \upsilon^{(5)}(I)$$

are automatically satisfied.

From the elimination ideal, we know that on solutions of the Euler-Lagrange equations the invariants $\upsilon^{(1)}(I)$ and $\upsilon^{(4)}(I)$ remain free.

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