# *VARIATIONAL BICOMPLEXES, GROUP ACTIONS, AND COHOMOLOGY*

*Juha Pohjanpelto*

Oregon State University

**Example.** Integrable systems.

*Potential Kadomtsev-Petviashvili* (PKP) equation

$$u_{tx} + \frac{3}{2} u_x u_{xx} + \frac{1}{4} u_{xxxx} + \frac{3}{4} s^2 u_{yy} = 0.$$

Admits an infinite dimensional algebra of distinguished symmetries $\mathfrak{g}_{PKP}$ involving 5 arbitrary functions of time $t$. (David, Kamran, Levi, Winternitz, *Symmetry reduction for the Kadomtsev-Petviashvili equation using a loop algebra,* J. Math. Phys. **27** (1986), 1225–1237.)

The symmetry algebra $\mathfrak{g}_{PKP}$ is spanned by the vector fields

$$X_f = f\,\frac{\partial}{\partial t} + \frac{2}{3}yf'\frac{\partial}{\partial y} + (\frac{1}{3}xf' - \frac{2}{9}s^2y^2f'')\,\frac{\partial}{\partial x} + (-\frac{1}{3}uf' + \frac{1}{9}x^2f'' - \frac{4}{27}s^2xy^2f''' + \frac{4}{243}y^4f'''')\,\frac{\partial}{\partial u},$$

$$Y_g = g\,\frac{\partial}{\partial y} - \frac{2}{3}s^2yg'\frac{\partial}{\partial x} + (-\frac{4}{9}s^2xyg'' + \frac{8}{81}y^3g''')\,\frac{\partial}{\partial u},$$

$$Z_h = h\,\frac{\partial}{\partial x} + (\frac{2}{3}xh' - \frac{4}{9}s^2y^2h'')\,\frac{\partial}{\partial u},$$

$$W_k = yk\,\frac{\partial}{\partial u}, \quad \text{and} \quad U_l = l\,\frac{\partial}{\partial u}\,,$$

where $f = f(t)$, $g = g(t)$, $h = h(t)$, $k = k(t)$ and $l = l(t)$ are arbitrary smooth functions of $t$.

Locally variational with the Lagrangian

$$L = -\frac{1}{2}u_t u_x - \frac{1}{4}u_x^3 + \frac{1}{8}u_{xx}^2 - \frac{3}{8}s^2 u_y^2.$$

But admits no Lagrangian for the PKP equation invariant under $\mathfrak{g}_{PKP}$!

To what extent do these properties characterize the PKP equation?

*Modified KdV* (MKdV) equation

$$v_t - v_{xxx} - v^2 v_x = 0.$$

Write $v = u_x$ to get the variationally closed equation

$$u_{tx} - u_{xxxx} - u_x^2 u_{xx} = 0 \qquad (\star)$$

with the Lagrangian

$$L = -\frac{1}{2} u_t u_x + \frac{1}{2} u_{xx}^2 - \frac{1}{12} u_x^4.$$

Equation $(\star)$ admits an infinite dimensional algebra $\mathfrak{g}_{MKdV}$ of distinguished symmetries with the generators

$$\mathbf{t}_t = \frac{\partial}{\partial t}, \quad \mathbf{t}_x = \frac{\partial}{\partial x}, \quad \mathbf{s} = t\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x}, \quad \mathbf{u}_f = f(t)\frac{\partial}{\partial u},$$

where $f(t)$ is an arbitrary function of time.

But admits no Lagrangian for $(\star)$ invariant under $\mathfrak{g}_{MKdV}$!

*Davey-Stewartson* (DS) equations

$$
\begin{aligned}
-v_t + a(u_{xx} + u_{yy}) - bu(u^2 + v^2) - cuw_y = 0,\\
u_t + a(v_{xx} + v_{yy}) - bv(u^2 + v^2) - cvw_y = 0,\\
w_{xx} - w_{yy} + c(uu_y + vv_y) = 0.
\end{aligned}
$$

Admits an infinite dimensional algebra of distinguished symmetries $\mathfrak{g}_{DS}$ involving 6 arbitrary functions of time $t$. (Omote, M., *Infinite dimensional symmetry algebras and an infinite number of conserved quantities of the (2+1)-dimensional Davey-Stewartson equation,* J. Math. Phys. **29** (1988), 2599–2603.)

The Lagrangian

$$
L = vu_t - \frac{a}{2}(u_x^2 + u_y^2 + v_x^2 + v_y^2) - \frac{b}{4}(u^2 + v^2)^2 - \frac{c}{2}(u^2 + v^2)w_y - w_x^2 + w_y^2
$$

is not invariant under $\mathfrak{g}_{DS}$.

**Example.** Gauss-Bonnet Theorem.

Even dimensional compact orientable Riemannian manifold $(M^{2m}, g)$ with curvature form $\Omega$.

*Euler form* $Pf(\Omega)$;

*Euler characteristic* $\mathcal{X}(M)$.

*Gauss-Bonnet*: $\int_M Pf(\Omega) = (2\pi)^m \mathcal{X}(M)$.

The Euler form is invariant under orientation preserving diffeomorphisms and its Euler-Lagrange expression vanishes.

How does one identify other analogous objects on Riemannian manifolds?

**Example.** Foliations.

Codimension $q$ transversally orientable integrable distribution $\Delta$ on $M$.

Frobenius: There is $\omega \in \Omega^q$ so that

$$X \in \Gamma(\Delta) \quad \text{iff} \quad X \lrcorner\, \omega \equiv 0.$$

Integrability of $\Delta \quad \Leftrightarrow \quad d\omega = \eta \wedge \omega$.

Godbillon-Vey form: $\gamma = \eta \wedge (d\eta)^q$.

The *Godbillon-Vey class* is the cohomology class of $[\gamma] \in H^{2q+1}(M)$.

Proper framework, generalizations, extensions?

Goal is to reduce these type of questions to algebraic problems

## Diffeomorphism Pseudogroup

| | |
|---|---|
| $M^m$ | $m$ dimensional manifold |
| $\mathcal{D} = \mathcal{D}(M)$ | pseudogroup of local diffeomorphisms of $M$ |
| $\mathcal{D}^n \subset J^n(M, M)$ | bundle of $n$th order jets, $0 \leq n \leq \infty$ |

*Coordinates on* $\mathcal{D}^n$:

$$g^n = j_z^n \varphi = (z^a, Z^b, Z^b_{c_1}, Z^b_{c_1 c_2}, \ldots),$$

where $z^a$, $Z^b$ are local coordinates of $M$ about the source and the target, and $Z^b_{c_1}$, $Z^b_{c_1 c_2}$, $\ldots$ stand for the derivative variables.

## Pseudogroups

$\mathcal{G} \subset \mathcal{D}$ is a *pseudogroup* if

1. $\mathrm{id} \in \mathcal{G}$;
2. $\varphi, \psi \in \mathcal{G} \Rightarrow \varphi \circ \psi \in \mathcal{G}$ where defined;
3. $\varphi \in \mathcal{G} \Rightarrow \varphi^{-1} \in \mathcal{G}$.

$\mathcal{G}$ is a *Lie* (or continuous) *pseudogroup* if, in addition, for all $n \geq N$,

4. $\mathcal{G}^n \subset \mathcal{D}^n$ is a subbundle;
5. $\varphi \in \mathcal{G} \Longleftrightarrow j_z^n \varphi \in \mathcal{G}^n$;
6. $\mathcal{G}^{N+k} = \mathrm{pr}^k\, \mathcal{G}^N, \quad k \geq 1$.

## Infinitesimal generators

A local vector field $\mathbf{v} \in \mathcal{X}(M)$ is a $\mathcal{G}$ *vector field*, $\mathbf{v} \in \mathfrak{g}$, if the flow $\Phi_t^{\mathbf{v}} \in \mathcal{G}$ for all fixed $t$ on some interval about 0.

Let $\mathcal{G}^n$ be determined locally by $F_\alpha(z, Z^{(n)}) = 0$. Then a $\mathcal{G}$ vector field $\mathbf{v}$ satisfies

$$F_\alpha(z, \Phi_t^{\mathbf{v}\,(n)}) = 0 \quad \Longrightarrow \quad L_\alpha(z, j_z^n \mathbf{v}) = 0.$$

These are the *infinitesimal determining equations for* $\mathcal{G}$.

Maurer–Cartan forms for $\mathcal{D}^\infty$:

$\mathcal{D}$ acts on $\mathcal{D}^n$ from both left and right by

$$L_\psi j^n_z \varphi = j^n_z(\psi \circ \varphi),$$
$$R_\psi j^n_z \varphi = j^n_{\psi^{-1}(z)}(\varphi \circ \psi).$$

*Horizontal forms*: $dz^a$

*Basic Contact forms*: $\theta^b_{c_1 \cdots c_p} = dZ^a_{c_1 \cdots c_p} - \sum_i^m Z^a_{c_1 \cdots c_p c_{p+1}} dz^{c_{p+1}}$

*Maurer-Cartan forms* are represented by invariant contact forms on $\mathcal{D}^\infty$.

Construction of right invariant forms on $\mathcal{D}^\infty$:

The target coordinate $Z^b$ invariant under $R_\psi \Longrightarrow$

$$\omega^b = d_H Z^b = \textstyle\sum_c Z^b_c dz^c, \\ \mu^b = d_V Z^b = dZ^b - Z^b_c dz^c,$$

are also invariant under $R_\psi$.

Operators of invariant differentiation:

$$\mathbb{D}_{Z^a} = W^b_a \mathbb{D}_{z^b}, \quad \text{where}$$

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \textstyle\sum_{p\geq 0} Z^c_{d_1\cdots d_p b} \dfrac{\partial}{\partial Z^c_{d_1\cdots d_p}} \quad \text{and} \quad W^a_b = (Z^{-1})^a_b.$$

Right invariant coframe on $\mathcal{D}^\infty$:

$$\omega^a, \qquad \mu^a_{b_1\cdots b_p} = \mathcal{L}_{\mathbb{D}_{Z^{b_1}}} \cdots \mathcal{L}_{\mathbb{D}_{Z^{b_p}}} \mu^a, \quad p \geq 0.$$

Structure Equations:

Taylor series method: Let

$$\mu^a[\![H]\!] = \sum_{|J|\geq 0} \frac{1}{J!} \mu^a_J H^J,$$

where $J = (j_1, j_2, \ldots, j_k)$, $1 \leq j_l \leq m$, is a multi-index of length $|J| = k$.

Write

$$\mu[\![H]\!] = (\mu^1[\![H]\!], \ldots, \mu^m[\![H]\!])^T, \qquad \omega = (\omega^1, \ldots, \omega^m)^T.$$

Then
$$\begin{aligned} d\mu[\![H]\!] &= \nabla_H \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ), \\ d\omega &= -\omega \wedge \nabla_H \mu[\![0]\!]. \end{aligned}$$

Invariant coframe for $\mathcal{G}^\infty$: Simply pull back $\omega^i$, $\mu^a_J$ to $\mathcal{G}^\infty$.

Relations: On $\mathcal{G}^\infty$, the Maurer-Cartan forms $\mu^a_J$ satisfy the right-invariant infinitesimal determining equations

$$L_\alpha(Z, \mu^a_J) = 0. \qquad (1)$$

Structure equations for $\mathcal{G}^\infty \backsim$ structure equations for $\mathcal{D}^\infty$ modulo relations (1).

**Moving frames.** Let $\pi\colon E \to M$ be a fibered manifold and write $J^n(E)$, $0 \le n \le \infty$, for the $n$th order jet bundle of local sections of $E \to M$.

Locally $J^n(E) \sim \bigl\{(x^i, u^\alpha, u^\alpha_{i_1}, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_{i_1 i_2 \cdots i_n})\bigr\}$.

The action of $\mathcal{D}$ on $E$ lifts to $J^n$ through its action on sections, and this action factors into an action of $\mathcal{D}^n$ (and $\mathcal{G}^n$, $\mathfrak{g}$, … ) on $J^n$. This process is called *prolongation*.

A local *moving frame* of order $n$ is a $\mathcal{G}$-equivariant mapping

$$\rho^n\colon \mathcal{V} \to \mathcal{G}^n, \qquad \mathcal{V}^{\text{open}} \subset J^n,$$

preserving base points.

Existence of moving frames.

*Isotropy subgroup* at $z^n$:

$$\mathcal{I}^n_{z^n} = \{ g^n \in \mathcal{G}^n_z \mid g^n \cdot z^n = z^n \}.$$

$\mathcal{G}$ *acts freely* at $z^n$ if $\mathcal{I}^n_{z^n} = \{\mathrm{id}^n_z\}$ and *locally freely* at $z^n$ if $\mathcal{I}^n_{z^n}$ is a discrete subgroup of $\mathcal{G}^n_z$.

**Theorem.** *A local moving frame of order $n$ exists in a neighborhood of $z^n \in J^n$ if and only if $\mathcal{G}$ acts locally freely at $z^n$.*

**Theorem.** *If $\mathcal{G}^n$ acts (locally) freely at $z^n \in J^n$, then $\mathcal{G}^l$ acts (locally) freely at any $z^l \in J^l$ with $\pi^l_n(z^l) = z^n$, for $l > n$.*

**Theorem.** *Suppose $\mathcal{G}^n$ admits a moving frame on $\mathcal{V} \subset J^n$. Then $\mathcal{G}^{n+1}$ admits a moving frame on $\mathcal{V}^{(1)} = (\pi^{n+1}_n)^{-1}(\mathcal{V})$.*

$\Longrightarrow$ The projective limit of a compatible sequence of moving frames is well defined on $J^\infty$.

Construction: Choose a cross-section $\mathcal{K}$ for the action of $\mathcal{G}^n$ on $J^n$. Define $\rho^n(z^n)$ by the condition $\rho^n(z^n) \cdot z^n \in \mathcal{K}$.

Opposite view: Let

$\mathcal{H}^n_{|\mathcal{K}} = \{(g^n, z^n) \mid z^n \in \mathcal{K},\ g^n,\ z^n \text{ based at the same point}\}$,

and let

$$\tau^n \colon \mathcal{H}^n_{|\mathcal{K}} \to J^n, \quad \tau^n(g^n, z^n) = g^n \cdot z^n.$$

Then, if the action is locally free, $\tau^n$ will be an equivariant local diffeomorphism with the action of $\mathcal{G}$ on $\mathcal{H}^n_{|\mathcal{K}}$ given by $\varphi \cdot (g^n, z^n) = (\varphi \cdot g^n, z^n)$.

**Invariantization.** Let $\mathcal{H}^n$ be the pull-back of $\mathcal{G}^n \to M$ under $\widetilde{\pi}_o^n \colon J^n \to M$; hence $\mathbf{g}^n \in \mathcal{H}^n$ is the pair

$$\mathbf{g}^n = (z^n, g^n),$$

where both $z^n \in J^n$, $g^n \in \mathcal{G}^n$ are based at the same point $z \in M$.

Source and target maps

$$\sigma^n(\mathbf{g}^n) = z^n, \quad \tau^n(\mathbf{g}^n) = g^n \cdot z^n.$$

$\mathcal{G}$ acts on $\mathcal{H}^n$ from the left by

$$L_\psi \mathbf{g}^n = (j_z^n \psi \cdot z^n, g^n \cdot j_{\psi(z)}^n \psi^{-1}).$$

Then $\tau^n(L_\psi \mathbf{g}^n) = \tau^n(\mathbf{g}^n)$ so that the target coordinates are $\mathcal{G}$ invariant.

Decompose $\Omega^*(\mathcal{H}^\infty) = \oplus_{i,j,k}\Omega^{i,j,k}$, where

| | |
|---|---|
| $i$ | is the horizontal degree in $J^\infty$ |
| $j$ | is the contact degree in $J^\infty$ |
| $k$ | is the contact degree in $\mathcal{G}^\infty$ |

Let $\pi_J$ be the projection $\pi_J \colon \Omega^* \to \oplus_{i,j}\Omega^{i,j,0}$.
$\pi_J$ preserves $\mathcal{G}$ invariance.

The *lift* of $\omega \in \Omega^*(J^\infty)$ is $\lambda(\omega) = \pi_J(\tau^*(\omega))$.

Given a moving frame $\rho$ for a pseudogroup $\mathcal{G}$, the *invariantization* $\iota(\omega)$ of $\omega$ is

$$\iota(\omega) = \rho^*\lambda(\omega).$$

**Theorem.** *The invariantization of a local coframe on* $J^\infty$ *produces an* $\mathcal{G}$-invariant *local coframe on* $J^\infty$*.*

Variational Bicomplex

The cotangent bundle to $J^\infty(E)$ splits

$$\Omega^*(J^\infty(E)) = \sum_{r,s\geq 0} \Omega^{r,s}(J^\infty(E)),$$

where $\omega \in \Omega^{r,s}(J^\infty(E))$ is a finite sum of terms of the form

$$f(x^i, u^\alpha, u^\alpha_i, \ldots, u^\alpha_{i_1\cdots i_k}) dx^{j_1} \wedge \cdots \wedge dx^{j_r} \wedge \theta^{\alpha_1}_{J_1} \wedge \cdots \wedge \theta^{\alpha_s}_{J_s}.$$

$(\theta^\alpha_J = du^\alpha_J - u^\alpha_{Jj} dx^j)$

Coordinate *total derivative operators*

$$\begin{aligned} D_i &= \frac{\partial}{\partial x^i} + \sum_{|I|\geq 0} u^\alpha_{Ii} \frac{\partial}{\partial u^\alpha_I} \\ &= \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + u^\alpha_{ijk} \frac{\partial}{\partial u^\alpha_{jk}} + \cdots \end{aligned}$$

span the module of *horizontal vector fields*, which gives rise to a flat connection on $J^\infty(E) \to M$

Splitting of the exterior derivative:

$$
\begin{aligned}
&d = d_H + d_V, && d^2 = 0, \\
&d_H : \Omega^{r,s} \to \Omega^{r+1,s}, && d_H^2 = 0, \\
&d_V : \Omega^{r,s} \to \Omega^{r,s+1}, && d_V^2 = 0, \\
&d_H d_V + d_V d_H = 0.
\end{aligned}
$$

Get the double complex

$$
\begin{array}{ccccccccccc}
 & & \uparrow d_V & & & & & & & & \uparrow d_V \\
0 & \to & \Omega^{0,3} & & & & \cdots & & & & \Omega^{m,3} \\
 & & \uparrow d_V & & & & & & & & \uparrow d_V \\
0 & \to & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \Omega^{m-1,2} & \xrightarrow{d_H} & & \Omega^{m,2} \\
 & & \uparrow d_V & & \uparrow d_V & & & \uparrow d_V & & & \uparrow d_V \\
0 & \to & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \Omega^{m-1,1} & \xrightarrow{d_H} & & \Omega^{m,1} \\
 & & \uparrow d_V & & \uparrow d_V & & & \uparrow d_V & & & \uparrow d_V \\
\mathbb{R} & \to & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \Omega^{m-1,0} & \xrightarrow{d_H} & & \Omega^{m,0} \\
 & & \uparrow \pi^* & & \uparrow \pi^* & & & \uparrow \pi^* & & & \uparrow \pi^* \\
 & & \Omega^0_M & \xrightarrow{d} & \Omega^1_M & \xrightarrow{d} & \cdots & \Omega^{m-1}_M & \xrightarrow{d} & & \Omega^m_M
\end{array}
$$

Define

$$\partial_\alpha^I u_J^\beta = \begin{cases} \delta_\alpha^\beta \delta_{j_1}^{(i_1} \cdots \delta_{j_p}^{i_p)}, & \text{if } |I| = |J|, \\ 0 \quad \text{otherwise.} \end{cases}$$

*Interior Euler operators* $F_\alpha^I \colon \Omega^{r,s} \to \Omega^{r,s-1}$, $s \geq 1$,

$$F_\alpha^I(\omega) = \sum_{|J| \geq 0} \binom{|I| + |J|}{|I|} (-D)_J (\partial_\alpha^{IJ} \lrcorner\, \omega).$$

*Integration-by-parts operator*

$I : \Omega^{m,s}(J^\infty(E)) \to \mathcal{F}^s(J^\infty(E)) \subset \Omega^{m,s}(J^\infty(E)), \quad s \geq 1,$

$$I(\omega) = \frac{1}{s}\, \theta^\alpha \wedge F_\alpha(\omega).$$

Spaces of *functional* $s$*-forms* $\mathcal{F}^s = I(\Omega^{m,s})$, $s \geq 1$.

Differentials $\quad \delta_V = I \circ d_V \colon \mathcal{F}^s \to \mathcal{F}^{s+1}$.

Then $\quad \delta_V^2 = 0$.

*Free Variational Bicomplex* (Anderson, Gelfand, Tsujishita, Tulczyjew, Vinogradov)

$$
\begin{array}{ccccccccccc}
 & & \uparrow{\scriptstyle d_V} & & & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} \\
0 & \to & \Omega^{0,3} & & & & \cdots & & \Omega^{m,3} & \xrightarrow{I} & \mathcal{F}^3 \\
 & & \uparrow{\scriptstyle d_V} & & & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} \\
0 & \to & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,2} & \xrightarrow{d_H} & \Omega^{m,2} & \xrightarrow{I} & \mathcal{F}^2 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta_V} \\
0 & \to & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,1} & \xrightarrow{d_H} & \Omega^{m,1} & \xrightarrow{I} & \mathcal{F}^1 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \\
\mathbb{R} & \to & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,0} & \xrightarrow{d_H} & \Omega^{m,0} & & \\
 & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \\
 & & \Omega^0_M & \xrightarrow{d} & \Omega^1_M & \xrightarrow{d} & \cdots\ \Omega^{m-1}_M & \xrightarrow{d} & \Omega^m_M & &
\end{array}
$$

The edge complex

$$\mathbb{R} \to \Omega^{0,0} \xrightarrow{d_H} \Omega^{1,0} \xrightarrow{d_H} \cdots$$

$$\xrightarrow{d_H} \Omega^{m-1,0} \xrightarrow[\text{Div}]{d_H} \Omega^{m,0} \xrightarrow[\text{E}]{\delta_V} \mathcal{F}^1 \xrightarrow[\mathcal{H}]{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots$$

is called the *Euler-Lagrange* complex $\mathcal{E}^*(J^\infty(E))$.

Both the variational bicomplex and the Euler-Lagrange complex are *locally exact*.

Horizontal homotopy operator

$$h_H^{r,s}(\omega) = \frac{1}{s} \sum_{|I|\geq 0} c_I D_I \big[\theta^\alpha \wedge F_\alpha^{Ij}(D_j \lrcorner\, \omega)\big], \quad s \geq 1,$$

where $c_I = \frac{|I|+1}{n-r+|I|+1}$. Thus

$$\omega = h_H^{r+1,s}(d_H\omega) + d_H h_H^{r,s}(\omega), \quad r \leq m-1;$$
$$\omega = I(\omega) + d_H h_H^{m,s}(\omega), \quad r = m.$$

**Example:** $E = \{(t,x,y,u)\} \to \{(t,x,y)\}$.

$$\omega^1 = V_1 dt + V_2 dx + V_3 dy \in \Omega^{1,0}$$
$$\Longleftrightarrow \text{a vector field} \quad V = (V_1, V_2, V_3),$$
$$\omega^2 = W_1 dx \wedge dy + W_2 dy \wedge dt + W_3 dt \wedge dx \in \Omega^{2,0}$$
$$\Longleftrightarrow \text{a vector field} \quad W = (W_1, W_2, W_3),$$
$$\omega^3 = L dt \wedge dx \wedge dy \in \Omega^{3,0}$$
$$\Longleftrightarrow \text{a Lagrangian } L,$$
$$\omega^4 = \Delta\, \theta \wedge dt \wedge dx \wedge dy \in \mathcal{F}^1$$
$$\Longleftrightarrow \text{a differential equation } \Delta = 0.$$

$(A)$ $\omega^2 = d_H \omega^1 \quad \Longleftrightarrow \quad W_1 = D_x V_2 - D_y V_3,$
$W_2 = D_y V_1 - D_t V_3, \quad W_3 = D_t V_2 - D_x V_1;$

$(B)$ $\omega^3 = d_H \omega^2 \quad \Longleftrightarrow \quad L = D_t W_1 + D_x W_2 + D_y W_3;$

$(C)$ $\omega^4 = \delta_V \omega^3 \quad \Longleftrightarrow$

$$\Delta = \mathrm{E}(L) = \frac{\partial L}{\partial u} - D_t\Big(\frac{\partial L}{\partial u_t}\Big) - D_x\Big(\frac{\partial L}{\partial u_x}\Big) + D_t D_t\Big(\frac{\partial L}{\partial u_{tt}}\Big) + \cdots .$$

Moreover,

$$(D) \quad \delta_V \omega^4 = -\frac{1}{2} \mathcal{H}^{pqr}_{\Delta} \theta \wedge \theta_{t^p x^q y^r} \wedge dt \wedge dx \wedge dy \quad \text{with}$$
$$\mathcal{H}^{pqr}_{\Delta} = \frac{\partial \Delta}{\partial u_{t^p x^q y^r}} - (-1)^{p+q+r} \mathrm{E}^{pqr}(\Delta),$$

where

$$\mathrm{E}^{pqr}(\Delta) = \frac{\partial \Delta}{\partial u_{t^p x^q y^r}} - (p+q+r+1) D_t \left( \frac{\partial \Delta}{\partial u_{t^{p+1} x^q y^r}} \right) + \cdots .$$

The complex

$$\mathbb{R} \to \Omega^{0,0} \xrightarrow[\text{Grad}]{d_H} \Omega^{1,0}$$
$$\xrightarrow[\text{Curl}]{d_H} \Omega^{2,0} \xrightarrow[\text{Div}]{d_H} \Omega^{3,0} \xrightarrow[\text{E}]{\delta_V} \mathcal{F}^1 \xrightarrow[\text{H}]{\delta_V} \mathcal{F}^2 \xrightarrow{\delta_V} \cdots$$

is locally exact.

Next let $\mathcal{G}$ be a pseudogroup of *projectable* transformations acting on $E$. By restricting the variational bicomplex to $\mathrm{pr}\,\mathcal{G}$ invariant elements one gets the $\mathcal{G}$ *invariant variational bicomplex* $\Omega^{*,*}_{\mathcal{G}}(J^\infty(E))$:

$$\begin{array}{ccccccccccc}
 & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V \\
0 & \rightarrow & \Omega^{0,3}_{\mathcal{G}} & & & & \cdots & & \Omega^{m,3}_{\mathcal{G}} & \xrightarrow{I} & \mathcal{F}^3_{\mathcal{G}} \\
 & & \uparrow d_V & & & & & & \uparrow d_V & & \uparrow \delta_V \\
0 & \rightarrow & \Omega^{0,2}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{1,2}_{\mathcal{G}} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,2}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{m,2}_{\mathcal{G}} & \xrightarrow{I} & \mathcal{F}^2_{\mathcal{G}} \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow \delta_V \\
0 & \rightarrow & \Omega^{0,1}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{1,1}_{\mathcal{G}} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,1}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{m,1}_{\mathcal{G}} & \xrightarrow{I} & \mathcal{F}^1_{\mathcal{G}} \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \\
\mathbb{R} & \rightarrow & \Omega^{0,0}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{1,0}_{\mathcal{G}} & \xrightarrow{d_H} & \cdots\ \Omega^{m-1,0}_{\mathcal{G}} & \xrightarrow{d_H} & \Omega^{m,0}_{\mathcal{G}} & &
\end{array}$$

The edge complex

$$\mathbb{R} \to \Omega^{0,0}_{\mathcal{G}} \xrightarrow{d_H} \Omega^{1,0}_{\mathcal{G}} \xrightarrow{d_H} \cdots$$

$$\xrightarrow{d_H} \Omega^{m-1,0}_{\mathcal{G}} \xrightarrow{d_H} \Omega^{m,0}_{\mathcal{G}} \xrightarrow{\mathrm{E}} \mathcal{F}^1_{\mathcal{G}} \xrightarrow{\mathrm{H}} \mathcal{F}^2_{\mathcal{G}} \xrightarrow{\delta_V} \cdots$$

is the $\mathcal{G}$ *invariant Euler-Lagrange complex* $\mathcal{E}^*_{\mathcal{G}}(J^\infty(E))$.

Associated cohomology spaces:

$$H^r(\mathcal{E}^*_{\mathcal{G}}(J^\infty(E))) = \frac{\ker \delta_V : \mathcal{E}^r_{\mathcal{G}} \to \mathcal{E}^{r+1}_{\mathcal{G}}}{\mathrm{im}\ \delta_V : \mathcal{E}^{r-1}_{\mathcal{G}} \to \mathcal{E}^r_{\mathcal{G}}}\,.$$

Example: Integrable systems again.

PKP equation:

*Bundle*: $E = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$;
*Group*: symmetry group $\mathfrak{g}_{PKP}$ of the PKP equation.

The *PKP source form*

$$\Delta_{PKP} = \left( u_{tx} + \frac{3}{2} u_x u_{xx} + u_{xxxx} + \frac{3}{4} s^2 u_{yy} \right) \theta \wedge dt \wedge dx \wedge dy$$

generates non-trivial cohomology in $H^4(\mathcal{E}_{\mathfrak{g}_{PKP}}(J^\infty(E)))$.

Compute $H^*(\mathcal{E}_{\mathfrak{g}_{PKP}}(J^\infty(E)))$!

MKdV equation:

*Bundle*: $E = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$;
*Group*: symmetry group $\mathfrak{g}_{\mathrm{MKdV}}$ of the MKdV equation.

The *MKdV source form*

$$\Delta_{MKdV} = \big(u_{tx} - u_{xxxx} - u_x^2 u_{xx}\big)\, \theta \wedge dt \wedge dx$$

generates non-trivial cohomology in $H^3(\mathcal{E}_{\mathfrak{g}_{\mathrm{MKdV}}}(J^\infty(E)))$.

Compute $H^*(\mathcal{E}_{\mathcal{G}_{\mathrm{MKdV}}}(J^\infty(E)))$!

Davey-Stewartson Equation:

*Bundle*: $E = \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$;
*Group*: symmetry group $\mathfrak{g}_{\mathrm{DS}}$ of the DS equation.

Compute $H^*(\mathcal{E}_{\mathfrak{g}_{\mathrm{DS}}}(J^\infty(E)))$!

## Natural Variational Bicomplex for Riemannian Metrics

*Bundle*: $E = \mathbb{R}^m \times \mathbb{Q}^m \to \mathbb{R}^m$;
*Group*: $\mathfrak{g} =$ lift of $\mathcal{X}(\mathbb{R}^m)$ to $E$.

Gilkey: $H^p(\mathcal{E}_{\mathfrak{g}}(J^\infty(E))) = \begin{cases} \text{Pontrjagin classes, } p \le m; \\ \text{Euler class, } p = m = 2q. \end{cases}$

Anderson: $H^{m+1}(\mathcal{E}_{\mathfrak{g}}(J^\infty(E))) = \begin{cases} \{0\}, \quad m = 0, 1, 2 \pmod 4; \\ I(so(m)), \ m = 3 \pmod 4, \end{cases}$

where $I(so(m))$ denotes $so(m)$ invariant polynomials.

Compute $H^*(\mathcal{E}_{\mathfrak{g}}(J^\infty(E)))$!

FOLIATIONS

Codimension $q$ integrable distribution $\Delta$ is determined by $\omega^1, \ldots, \omega^q$ with $d\omega^a = \eta^a_b \wedge \omega^b$.

Model using the trivial foliation $\mathcal{F}_T = \{(x^i, y^a) \,|\, y^a = c^a\}$ of $\mathbb{R}^m$:

$M = \mathbb{R}^p \times \mathbb{R}^q = \{(x^i, y^a)\}$.

$P = Gl(q) \times M \to M, \qquad Q = \bigoplus_{q^2} T^*M \times M \to M.$

$E = P \oplus Q \to M; \quad E = \{(x^i, y^a, g^a_b, \eta^a_{b,i}, \eta^a_{b,c})\} \to \{(x^i, y^a)\}.$

$V \subset J^1(E)$ consists of all 1-jets satisfying $d\omega^a = \eta^a_b \wedge \omega^b$.

*Group:*

$$\mathfrak{g} = \left\{X \in \mathcal{X}(\mathbb{R}^m) \,|\, X = f^i(x^j, y^b)\frac{\partial}{\partial x^i} + g^a(y^b)\frac{\partial}{\partial y^a}\right\}$$

consist of vector fields on $\mathbb{R}^m$ preserving $\mathcal{F}_T$.

Compute $H^*(\mathcal{E}_{\mathfrak{g}}(V^\infty))$!

**Exactness of the Horizontal Rows.**

**Theorem.** *Let* $\mathfrak{g}$ *be a pseudgroup of projectable transformations acting on* $E \to M$*, and let* $\omega^i$ *and* $\Theta^\alpha$ *be* $\operatorname{pr}\mathfrak{g}$ *invariant horizontal frame and zeroth order contact frame defined on some open set* $\mathcal{U} \subset J^\infty(E)$ *contained in an adapted coordinate system.*

*Then the interior rows of the* $\operatorname{pr}\mathfrak{g}$ *invariant augmented variational bicomplex restricted to* $\mathcal{U}$ *are exact,*

$$H^*(\Omega^{*,s}_{\mathfrak{g}}(J^\infty(\mathcal{U}))) = \{0\}, \qquad s \geq 1.$$

**Corollary.** *Under the above hypothesis*

$$H^*(\mathcal{E}^*_{\mathfrak{g}}(\mathcal{U}), \delta_V) \simeq H^*(\Omega^*_{\mathfrak{g}}(\mathcal{U}), d).$$

The proof is based on the analysis of the highest weight terms in the contact forms:

Suppose

$$d_H\omega = 0 \quad \text{for } \omega \in \Omega_{\mathfrak{g}}^{r,s}, \quad s \geq 1.$$

Let $\mathbf{H}(\omega)$ consists of the highest weight terms in $\omega$ with the coefficients having been frozen. Then

$$d_H\mathbf{H}(\omega) = 0,$$

so the standard horizontal homotopy operator produces a form

$$\widehat{\eta} \in \Omega_{\Gamma}^{r-1,s} \quad \text{with} \quad d_H\widehat{\eta} = \mathbf{H}(\omega).$$

Next by unfreezing the coefficients in $\widehat{\eta}$ and doing some algebra, one can construct a $\mathfrak{g}$ invariant form $\eta$ so that the highest weight terms in $d_H\eta$ agree with $\mathbf{H}(\omega)$,

$$\mathbf{H}(d_H\eta) = \mathbf{H}(\omega).$$

Now proceed by induction.

The proof can be straightforwardly modified to prove the exactness of the interior rows of the invariant variational quasi-tricomplex (Kogan, Olver, 2001) for non-projectable pseudogroup actions.

**Computational Techniques.**

1. METHODS BASED ON THE EXPLICIT DESCRIPTION OF THE INVARIANT VARIATIONAL BICOMPLEX. Recall that the invariantization process yields complete sets of differential invariants and an invariant coframe.

$H^*(\Omega_{\mathfrak{g}}(\mathcal{U}), d)$ can be computed with the help of the *double complex spectral sequence* associated with the filtration

$$F^p\Omega_\Gamma = \oplus_{q\geq p}\Omega_\Gamma^{*,q}.$$

The first page of this consists of

$$H^{*,*}(\Omega_{\mathfrak{g}}(\mathcal{U}), d_V).$$

GELFAND-FUKS COHOMOLOGY: Formal power series vector fields on $\mathbb{R}^m$

$$W_m = \left\{ \sum_{i=1}^{m} a^l \frac{\partial}{\partial x^l} \mid a^l \in \mathbb{R}[[x^1, \ldots, x^m]] \right\}.$$

Lie bracket $[\ ,\ ]\colon W_m \times W_m \to W_m$.

Give $W_m$ a topology relative to the ideal $m = (x^1, \ldots, x^m)$.

$\Lambda_c^*(W_m)$: continuous alternating functionals on $W_m$.

$\Lambda_c^*(W_m)$ is generated by $\delta^i_{j_1 \cdots j_k}$, where

$$\delta^i_{j_1 \cdots j_k}(a^l \frac{\partial}{\partial x^l}) = \frac{\partial^k a^i}{\partial x^{j_1} \cdots \partial x^{j_k}}(0).$$

The differential $d : \Lambda_c^i(W_m) \to \Lambda_c^{i+1}(W_m)$ is induced by Lie bracket of vector fields so that

$$d\omega(X, Y) = -\omega([X, Y]).$$

$d^2 = 0!$

Let $\mathfrak{g}_o \subset \mathfrak{g} \subset W_m$ be subalgebras.

Define

$$\Lambda_c^*(\mathfrak{g}) = \Lambda_c^*(W_m)_{|\mathfrak{g}},$$
$$\Lambda_c^*(\mathfrak{g}, \mathfrak{g}_o) = \{\omega \in \Lambda_c^*(\mathfrak{g}) | X \lrcorner\, \omega = 0,\ X \lrcorner\, d\omega = 0, X \in \mathfrak{g}_o\}.$$

The *Gelfand-Fuks cohomology* $H^*(\mathfrak{g}, \mathfrak{g}_o)$ of $\mathfrak{g}$ relative to $\mathfrak{g}_o$ is the cohomology of the complex $(\Lambda_c^*(\mathfrak{g}, \mathfrak{g}_o), d)$.

If $\mathfrak{g}$ is finite dimensional, then $H^*(\mathfrak{g}, \mathfrak{g}_o)$ is the usual Lie algebra cohomology.

Evaluation Mapping: Pick $\sigma^\infty \in J^\infty(E)$.

For a given infinitesimal transformation group $\mathfrak{g}$ acting on $E$, let

$$\mathfrak{g}_o = \{X \in \mathfrak{g} \,|\, \mathrm{pr}\, X(\sigma^\infty) = 0\}.$$

Define $\rho\colon \Omega^*_{\mathfrak{g}}(J^\infty(E)) \to \Lambda^*_c(\mathfrak{g}, \mathfrak{g}_o)$ by

$$\rho(\omega)(X_1, \ldots, X_r) = (-1)^r \omega(\mathrm{pr}\, X_1, \ldots, \mathrm{pr}\, X_r)(\sigma^\infty).$$

Then $\rho$ is a cochain mapping, that is, commutes with the application of $d$, and thus induces a mapping

$$\rho^*\colon H^*(\Omega_{\mathfrak{g}}(J^\infty(E)), d) \to H^*(\mathfrak{g}, \mathfrak{g}_o).$$

2. Equivariant deformations

Construct a subbundle $\mathcal{K}^\infty \subset \mathcal{U} \subset J^\infty(E)$ with

(i) $\operatorname{pr}\mathfrak{g}$ acting transitively on $\mathcal{K}^\infty$; and

(ii) $\mathcal{K}^\infty$ is $\operatorname{pr}\mathfrak{g}$ equivariant strong deformation retract of $\mathcal{U}$, that is, there is a smooth map $H\colon \mathcal{U} \times [0,1] \to \mathcal{U}$ such that

$$
\begin{aligned}
H(\sigma^\infty, 0) &= \sigma && \text{for all } \sigma^\infty \in \mathcal{U},\\
H(\sigma^\infty, 1) &\in \mathcal{K}^\infty && \text{for all } \sigma^\infty \in \mathcal{U},\\
H(\sigma^\infty, t) &= \sigma^\infty && \text{for all } (\sigma^\infty, t) \in \mathcal{K}^\infty \times [0,1],\\
(H_t)_*(\operatorname{pr} X_{|\sigma^\infty}) &= \operatorname{pr} X_{|H(\sigma^\infty,t)} && \text{for all } X \in \mathfrak{g},\\
& && \quad (\sigma^\infty, t) \in \mathcal{U} \times [0,1].
\end{aligned}
$$

Under these conditions the inclusion map

$$\iota\colon \mathcal{K}^\infty \to \mathcal{U}$$

and the evaluation map

$$\varrho\colon \Omega^p_{\mathfrak{g}}(\mathcal{K}^\infty) \to \Lambda^p(\mathfrak{g}, \mathfrak{g}_o)$$

induce isomorphisms in cohomology

3. Moving Frames: Recall that

$$\tau^n \colon \mathcal{H}^n_{|\mathcal{K}} \to J^n$$

is an equivariant local diffeomorphism, so it suffices to compute $H^*(\Omega^*_{\mathcal{G}}(\mathcal{H}^n_{|\mathcal{K}}), d)$. Now the action of $\mathcal{G}$ on $\mathcal{K}$ is trivial, so by choosing $\mathcal{K}$ with trivial de Rham cohomology, the Künneth formula implies that

$$H^*(\mathcal{E}^*_{\mathcal{G}}(U), \delta_V) \simeq H^*(\Omega^*_{\mathcal{G}}(\mathcal{G}), d).$$

But $\Omega^*_{\mathcal{G}}(\mathcal{G})$ is effectively described by the infinitesimal determining equations.

Ideally the cohomology can be computed directly from the infinitesimal determining equations for $\mathcal{G}$.

**PKP equation again.** The symmetry group $\mathfrak{g}_{PKP}$ admits a basis $\alpha^n$, $\beta^n$, $\gamma^n$, $\upsilon^n$, $\vartheta^n$, $n \geq 0$, of invariant forms so that

$$d\alpha^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \wedge \alpha^{n-k+1},$$

$$d\beta^n = \sum_{k=0}^{n} \binom{n}{k} \big\{\alpha^k \wedge \beta^{n-k+1} - \frac{2}{3}\alpha^{k+1} \wedge \beta^{n-k}\big\},$$

$$d\gamma^n = \sum_{k=0}^{n} \binom{n}{k} \big\{\alpha^k \wedge \gamma^{n-k+1} - \frac{1}{3}\alpha^{k+1} \wedge \gamma^{n-k} - \frac{2}{3}s^2\beta^k \wedge \beta^{n-k+1}\big\},$$

$$d\upsilon^n = \sum_{k=0}^{n+1} \binom{n+1}{k} \big\{\alpha^k \wedge \upsilon^{n-k+1} + \frac{4}{9}s^2(\beta^{k+1} \wedge \gamma^{n-k+1} - 2\beta^k \wedge \gamma^{n-k+2})\big\},$$

$$d\vartheta^n = \sum_{k=0}^{n} \binom{n}{k} \big\{\alpha^k \wedge \vartheta^{n-k+1} + \frac{1}{3}\alpha^{k+1} \wedge \vartheta^{n-k} + \beta^k \wedge \upsilon^{n-k} + \frac{2}{3}\gamma^k \wedge \gamma^{n-k+1}\big\}.$$

Let $A$ be a non-vanishing differential function on some open set $\mathcal{U} \subset J^\infty(E)$ satisfying

$$\operatorname{pr} X_f(A) + \frac{1}{3} A f'(t) = 0, \qquad \frac{\partial A}{\partial y} = 0,$$

and let B be a differential function on $\mathcal{U}$ satisfying

$$\operatorname{pr} X_f(B) + \frac{2}{3} y A^{-1} f''(t) = 0, \qquad \frac{\partial B}{\partial y} = 0.$$

For example, one can choose

$$\begin{aligned} A &= (u_{x^n})^{\frac{1}{n+1}}, \qquad \text{and} \\ B &= -\frac{3}{2} s^2 u_{x^{n-1}y} (u_{x^n})^{-\frac{n+2}{n+1}}, \qquad n \geq 3. \end{aligned}$$

**Theorem.** *Suppose that differential functions $A$ and $B$, defined on an open $\mathcal{U} \subset J^\infty(E)$, are chosen as above. Then the dimensions of the cohomology spaces $H^p(\mathcal{E}^*_{\mathfrak{g}_{PKP}}(\mathcal{U}), \delta_V)$ are*

| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $p \geq 8$ |
|---|---|---|---|---|---|---|---|---|
| *dim* | 0 | 1 | 1 | 3 | 3 | 2 | 3 | 0 |

Let $\{\alpha^0, \beta^0, \gamma^0\}$ be the $\mathfrak{g}_{PKP}$ invariant horizontal frame defined by

$$\alpha^0 = A^3 dt, \qquad \beta^0 = A^2 dy + A^3 B dt,$$
$$\gamma^0 = A dx - \frac{2}{3} s^2 A^2 B dy + A^3 C dt,$$

where $C = -\frac{3}{2} u_x A^{-2} - \frac{1}{3} s^2 B^2$, and let $K$ be the $\mathfrak{g}_{PKP}$ differential invariant

$$K = (u_{tx} + \frac{3}{4} s^2 u_{yy} + \frac{3}{2} u_x u_{xx}) A^{-5}.$$

Moreover, let $\Delta^1$, $\Delta^2 \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be the source forms

$$\Delta^1 = (u_{tx} + \frac{3}{2}u_x u_{xx} + \frac{3}{4}s^2 u_{yy})\, dt \wedge dx \wedge dy \wedge du,$$
$$\Delta^2 = u_{xxxx}\, dt \wedge dx \wedge dy \wedge du,$$

and let $\Delta^3 \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be the source form which is the Euler-Lagrange expression

$$\Delta^3 = \mathrm{E}(BK\alpha^0 \wedge \beta^0 \wedge \gamma^0).$$

Note that the PKP source form is the sum

$$\Delta_{PKP} = \Delta^1 + \Delta^2.$$

**Corollary.** *Let $\Delta \in \mathcal{E}^4_{\mathfrak{g}_{PKP}}(\mathcal{U})$ be a $\mathfrak{g}_{PKP}$ invariant source form that is the Euler-Lagrange expression of some Lagrangian 3-form $\lambda \in \mathcal{E}^3(\mathcal{U})$. Then there are constants $c_1$, $c_2$, $c_3$ and a $\mathfrak{g}_{PKP}$ invariant Lagrangian 3-form $\lambda_0 \in \mathcal{E}^3_{\mathfrak{g}_{PKP}}(\mathcal{U})$ such that*

$$\Delta = c_1\Delta^1 + c_2\Delta^2 + c_3\Delta^3 + E(\lambda_0).$$

**Natural Variational Bicomplex of Riemannian Metrics.**

The coframe

$$\begin{aligned} &dx^i, \\ \theta_{ij} &= d_V g_{ij}, \\ \omega^i_{jk,l_1\cdots l_p} &= \nabla_{(l_1}\nabla_{l_2}\cdots\nabla_{l_p)} d_V \Gamma^i_{jk}, \\ \xi_{ij,l_1 l_2\cdots l_p} &= \nabla_{(l_1}\nabla_{l_2}\cdots\nabla_{l_p)} d_V R_i{}^k{}_{jk}, \end{aligned}$$

transforms homogeneously under $\mathfrak{g} = \mathcal{X}(\mathbb{R}^m) \Longrightarrow$ $\Omega^*_{\mathfrak{g}}(J^\infty(E))$ can be described explicitly.

Let

$$\begin{aligned} \omega^i_j &= g^{ik} d_V g_{jk}, \\ \gamma^i_j &= dx^k \wedge d_V \Gamma^i_{jk}, \\ \sigma^i_j &= \omega^i_k \wedge \omega^k_j. \end{aligned}$$

Use the bicomplex spectral sequence:
$H^{*,*}(\Omega_{\mathfrak{g}}(J^\infty(E)), d_V)$ is generated by

$$P(\omega^i_j), \quad Q(\gamma^i_j), \quad Pf(\sigma^i_j),$$

where $P$, $Q$ are $gl(m)$ invariant polynomials $\Longrightarrow$

$$H^*(\Omega_{\mathfrak{g}}(J^\infty(E)), d), \quad H^*(\mathcal{E}_{\mathfrak{g}}(J^\infty(E)), \delta_V)$$

can be computed by snaking.

Or, $J^\infty(E)$ can be deformed equivariantly to the bundle of flat metrics $\Longrightarrow$

$$H^*(\Omega_{\mathfrak{g}}(J^\infty(E))) \simeq H^*(\mathfrak{g}, so(m)).$$

*Truncated Weil algebra*: Start with

$$\begin{aligned} &c_1, c_2, \ldots, c_m, \\ &h_1, h_3, \ldots, h_{2k+1}, \quad 2k+1 \leq m < 2k+3, \end{aligned}$$

where

$$\begin{aligned} \deg c_i &= 2i, \qquad \deg h_i = 2i-1, \qquad \text{and} \\ dc_i &= 0, \qquad\qquad dh_i = c_i. \end{aligned}$$

$$\underline{WO}(m) = \frac{\Lambda(h_1, h_3, \ldots, h_{2k+1}) \otimes P(c_1, c_2, \ldots, c_m)}{\{\text{terms of degree} > 2m \text{ in } c_m\}}\,.$$

Bott, Häfliger:

$$H^*(\mathfrak{g}, so(m)) = \begin{cases} H^*(\underline{WO}(m)), \ m \text{ odd}; \\ H^*(\underline{WO})(m))[\Xi]/(\Xi^2 - c_m), \ m \text{ even}. \end{cases}$$

**Foliations.** Secondary characteristic classes:

$\mathfrak{h}^*$ the dual of a given finite dimensional Lie algebra $\mathfrak{h}$.

Weil algebra: $W(\mathfrak{h}) = \Lambda(\mathfrak{h}^*) \otimes S(\mathfrak{h}^*)$

Filtration $F_r W(\mathfrak{h}) = \bigoplus_{2j \geq r} \Lambda^*(\mathfrak{h}^*) \otimes S^j(\mathfrak{h}^*)$.

Differential

$$d_W(\gamma \otimes 1) = 1 \otimes \gamma + d_{\mathfrak{h}}\gamma \otimes 1,$$

where $d_{\mathfrak{h}}\gamma(v, w) = \gamma([v, w])$ for $\gamma \in \Lambda^1(\mathfrak{h})$.

Specialize to $\mathfrak{h} = gl(q)$, and define the *truncated Weil algebra* by

$$\underline{W}(gl(q)) = W(gl(q))/F_{2q+1}W(gl(q)).$$

Recall $d\omega^a = \eta^a_b \omega^b$. Define

$$\widehat{\eta}^a_b = \eta^a_b + \mu^a_c d_V \omega^c_b, \qquad \mu^a_b = (\omega^{-1})^a_b,$$

and write

$$\zeta^a_b = d\widehat{\eta}^a_c - \widehat{\eta}^a_c \wedge \widehat{\eta}^c_b.$$

Then $\zeta^{a_1}_{b_1} \wedge \cdots \wedge \zeta^{a_r}_{b_r} = 0$ for $r > q$. ($\rightsquigarrow$ Bott's vanishing theorem)

Let $e^a_b \in gl(q)^*$, $e^a_b(A) = A^a_b$.

Define

$$\Phi\colon \underline{W}(gl(q)) \to \Omega^*(V^\infty)$$

by

$$\Phi(e^a_b \otimes 1) = \widehat{\eta}^a_b, \qquad \Phi(1 \otimes e^a_b) = \zeta^a_b,$$

and extend multiplicatively. Then $\Phi$ induces a homomorphism

$$\overline{\Phi}\colon H^*(\underline{W}(gl(q))) \to H^*(\mathcal{E}_{\mathfrak{g}}(V^\infty)).$$

For example, $H^*(\underline{W}(gl(2)))$ is generated by

$$(\widehat{\eta}^1_1 + \widehat{\eta}^2_2) \wedge d(\widehat{\eta}^1_1 + \widehat{\eta}^2_2) \wedge d(\widehat{\eta}^1_1 + \widehat{\eta}^2_2);$$
$$(\zeta^1_1 \wedge \zeta^2_2 - \zeta^1_2 \wedge \zeta^2_1) \wedge (\widehat{\eta}^1_1 + \widehat{\eta}^2_2);$$
$$(\zeta^1_1 \wedge \zeta^2_2 - \zeta^1_2 \wedge \zeta^2_1) \wedge (\widehat{\eta}^1_1 - \widehat{\eta}^2_2) \wedge \widehat{\eta}^1_2 \wedge \widehat{\eta}^2_1;$$
$$d(\widehat{\eta}^1_1 + \widehat{\eta}^2_2) \wedge d(\widehat{\eta}^1_1 + \widehat{\eta}^2_2) \wedge \widehat{\eta}^1_1 \wedge \widehat{\eta}^1_2 \wedge \widehat{\eta}^2_1 \wedge \widehat{\eta}^2_2;$$
$$(\zeta^1_1 \wedge \zeta^2_2 - \zeta^1_2 \wedge \zeta^2_1) \wedge \widehat{\eta}^1_1 \wedge \widehat{\eta}^1_2 \wedge \widehat{\eta}^2_1 \wedge \widehat{\eta}^2_2;$$

Open question: Is $\overline{\Phi}$ surjective?