Moving Frames

— *since 1997*

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Moving Frames

Classical contributions:

M. Bartels (∼1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, . . .

The equivariant approach: (1997 –)

M. Fels & PJO, Moving coframes. I. A practical algorithm, Acta Appl. Math. **51** (1998) 161-213; II. Regularization and theoretical foundations, Acta Appl. Math. **55** (1999) 127-208.

E.L. Mansfield, *A Practical Guide to the Invariant Calculus*, Cambridge University Press, Cambridge, 2010

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
Bull. Amer. Math. Soc. **44** (1938) 598–601

moving frames $\neq$ frames

Equivariant Moving Frames

Definition.

A moving frame is a G-equivariant map (section)

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g\cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $G_z = \{ g \mid g \cdot z = z \}$ for $z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity

$$\Longrightarrow \quad G_z = \{e\} \text{ for all } z \in M$$

- locally free — the orbits all have the same dimension as G

$$\Longrightarrow \quad G_z \subset G \text{ is discrete for all } z \in M$$

- regular — the orbits form a regular foliation

$$\not\approx \quad \text{irrational flow on the torus}$$

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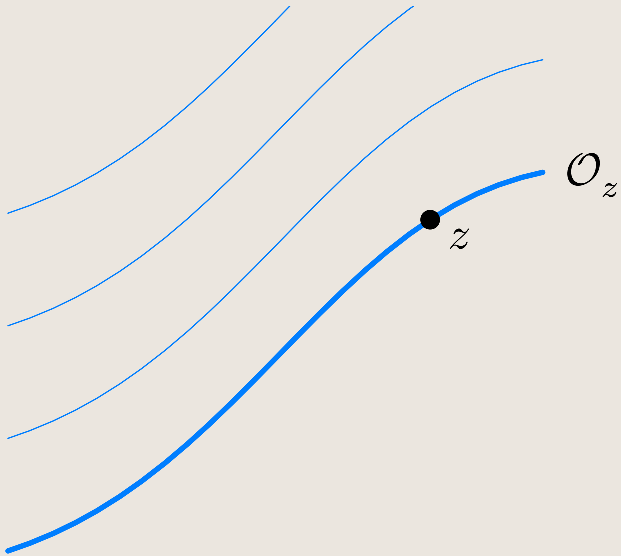
- effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

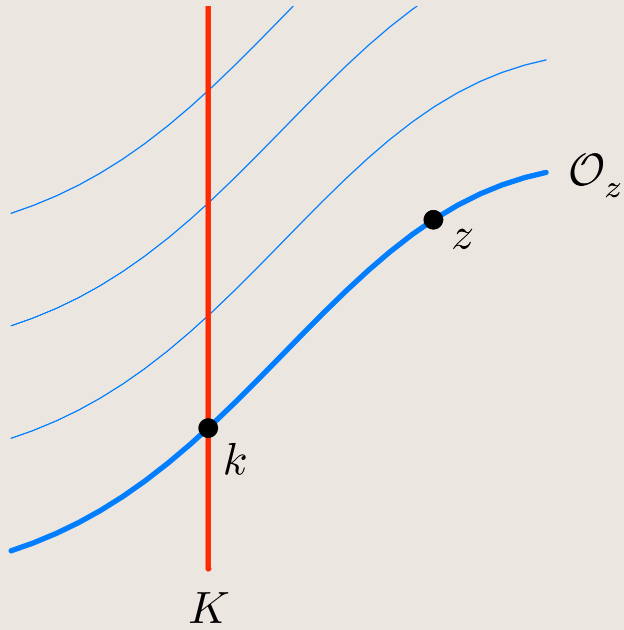
$$G_M^* = \bigcap_{z \in M} G_z = \{e\}$$

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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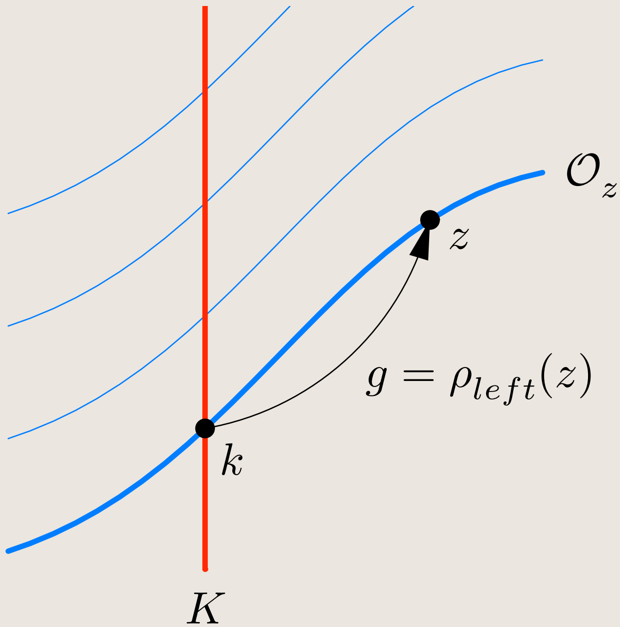
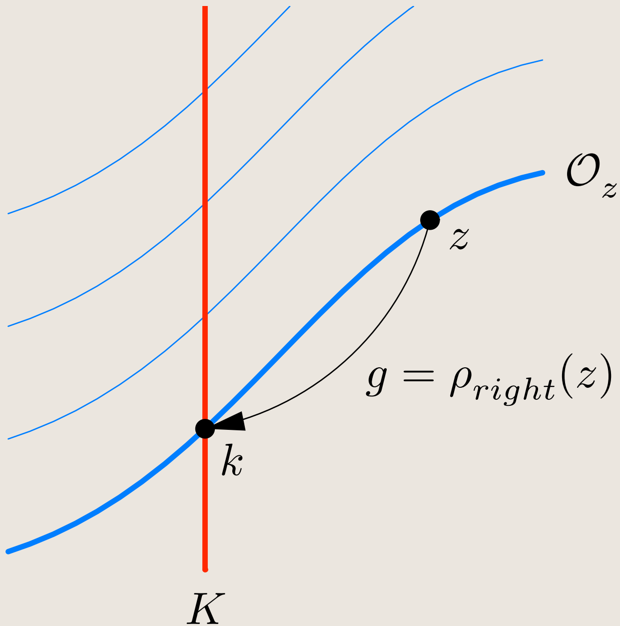


Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

Algebraic Construction

$$r = \dim G \;\le\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1, \quad \ldots \quad \iota(z_r) = c_r, \qquad \iota(z_{r+1}) = I_1(z), \quad \ldots \quad \iota(z_m) = I_{m-r}(z).$$

cross-section variables

"phantom invariants"

fundamental invariants

$$\boxed{\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \,|\, K = F \,|\, K$$

and then requiring $I = \iota(F)$ be constant on orbits.

Invariantization defines a canonical projection

$$\iota \; : \; \text{functions} \quad \longmapsto \quad \text{invariants}$$

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Rewrite Rule:

$$I(z_1, \ldots, z_m) \; = \; I(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

Action of $G = \mathrm{GL}(2)$ on $\mathbb{R}^2$ (or $\mathbb{C}^2$):

$$(x,u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}} = \frac{1}{n(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\, u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\, u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\, x u^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\, u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x + b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} = \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} = \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \quad \deg H = 2n-4 \quad \deg T = 3n-6 \quad \deg U = 4n-8$$

Differential Invariants

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged (pseudo-)group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(G)$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Recurrence Formulae

★ Invariantization and differentiation *do not commute*.

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

R_j^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Maurer–Cartan Invariants

R^κ_j — Maurer–Cartan invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual Maurer–Cartan forms

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R^\kappa_j \, \omega^j$$

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Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R^\kappa_j \, \omega^j$$

Remark: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants R^κ_j are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

The Maurer–Cartan Invariants

If the moving frame cross-section is given by

$$Z_1(x, u^{(n)}) = c_1, \quad \ldots \quad Z_r(x, u^{(n)}) = c_r,$$

then the Maurer–Cartan matrix $R = (\, R_i^\kappa \,)$ is given by

$$\boxed{R = -\,\iota[\, D(Z)\, \mathbf{v}(Z)^{-1}\,]}$$

where

$$D(Z) = (\, D_i Z_j \,), \qquad \mathbf{v}(Z) = (\, \mathbf{v}_\kappa^{(n)}(Z_i) \,).$$

Corollary. If the moving frame has order n, then the Maurer–Cartan invariants have order $\leq n + 1$.

The Commutator Invariants

Explicit formulae:

$$Y^i_{jk} = \sum_{\kappa=1}^{r} \sum_{j=1}^{p} R^\kappa_j \, \iota(D_j \xi^i_\kappa) - R^\kappa_k \, \iota(D_k \xi^i_\kappa) \,.$$

Follows from the recurrence formulae for

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= - \sum_{j<k} Y^i_{jk} \, \omega^j \wedge \omega^k + \cdots$$

Generating Differential Invariants

Theorem. (*Fels–O*) If the moving frame has order n, then the set of normalized differential invariants of order $\leq n+1$ forms a generating set.

Theorem. (*O–Hubert*) Given a *minimal order cross-section*, meaning that, for each $k = 0, 1, \ldots, n$,

$$Z_1(x, u^{(k)}) = c_1, \qquad \ldots \qquad Z_{r_k}(x, u^{(k)}) = c_{r_k},$$

defines a cross-section for the action of $G^{(k)}$ on J^k, then the differential invariants $\iota(D_i Z_j)$ for $i = 1, \ldots, p$, $j = 1, \ldots, r$ and, in the intransitive case, the order zero invariants, form a generating set.

Theorem. (*Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}(G)$.

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary = transitive + no pseudo-stabilization.

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* ordinary = transitive + no pseudo-stabilization.

$\Longrightarrow$ $m = 3$ — curvature κ & torsion τ

Equi-affine Surfaces

Theorem.

The algebra of equi-affine differential invariants for non-degenerate surfaces is generated by the Pick invariant through invariant differentiation.

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )$$

Euclidean Proof

Commutation relation:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Z_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2,$$

Commutator invariants:

$$Z_1 = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Z_2 = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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$$Z_1 = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Z_2 = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

Codazzi relation:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Z_1) Z_1 - (\mathcal{D}_2 + Z_2) Z_2$$

Euclidean Proof

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Commutator invariants:

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Codazzi relation:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Z_1) Z_1 - (\mathcal{D}_2 + Z_2) Z_2$$

$\Longrightarrow$ Gauss' Theorema Egregium

(*Guggenheimer*)

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H &= Z_2\,\mathcal{D}_1 H - Z_1\,\mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H &= Z_2\,\mathcal{D}_1\mathcal{D}_J H - Z_1\,\mathcal{D}_2\mathcal{D}_J H \end{aligned} \qquad (*)$$

Nondegenerate surface:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for Z_1, Z_2 in terms of derivatives of H.

Q.E.D.

Note: Any totally umbilic or constant mean curvature surface is degenerate. Are there others?

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \qquad \Longleftrightarrow \qquad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Equivalence & Syzygies

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

♡ The higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \longrightarrow \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\mathcal{S} = \operatorname{Im} \Sigma$$

is the signature subset (or classifying submanifold) of N.

Equivalence & Signature

Theorem. Two regular submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Other Signatures

Euclidean space curves: $\mathcal{C} \subset \mathbb{R}^3$

$$\mathcal{S} = \{ \, (\, \kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \quad \subset \quad \mathbb{R}^3$$

• κ — curvature, τ — torsion

Euclidean surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

• H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

• P — Pick invariant

Equivalence and Signature Curves

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{\, g \mid g \cdot N \subset N \,\}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N \;=\; \dim N - \dim \mathcal{S}$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \;\leq\; \dim G_N \;\leq\; \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders, planes, . .

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

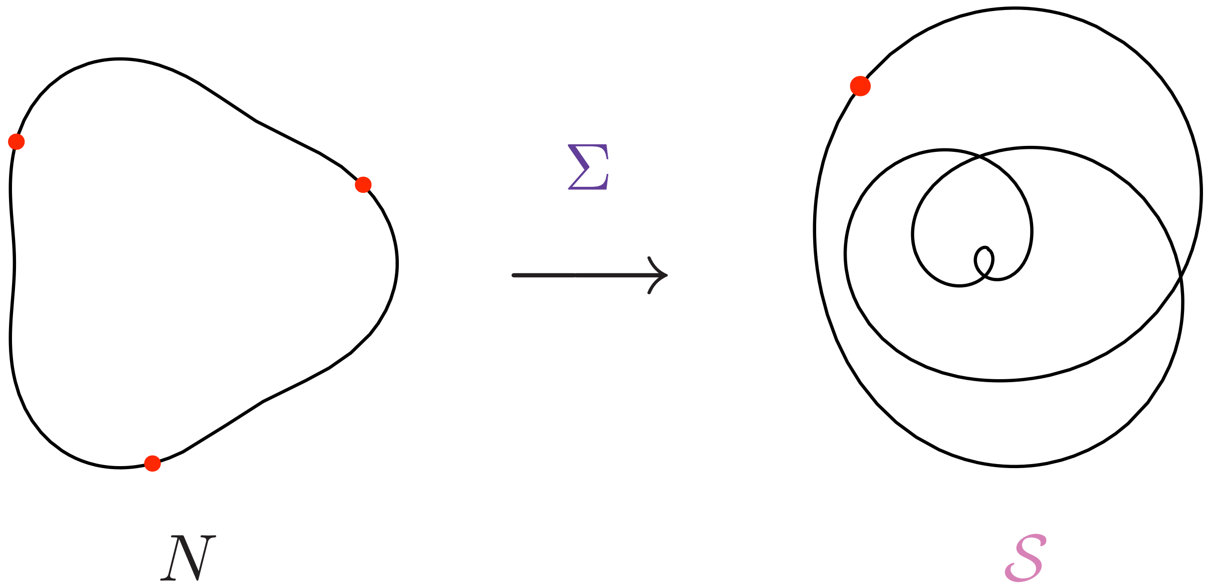
$$\iota_N = \min \left\{ \# \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

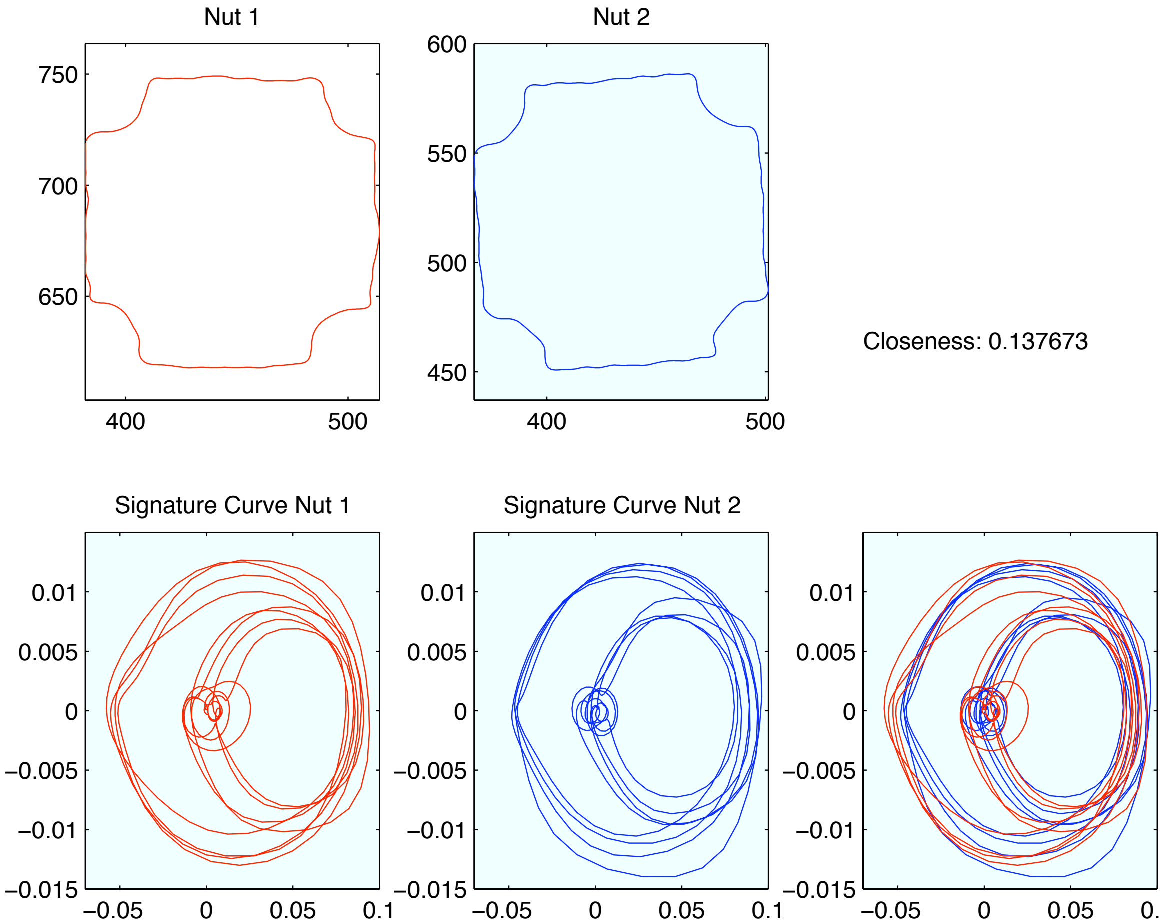
$\Longrightarrow$ Approximate symmetries

The Index

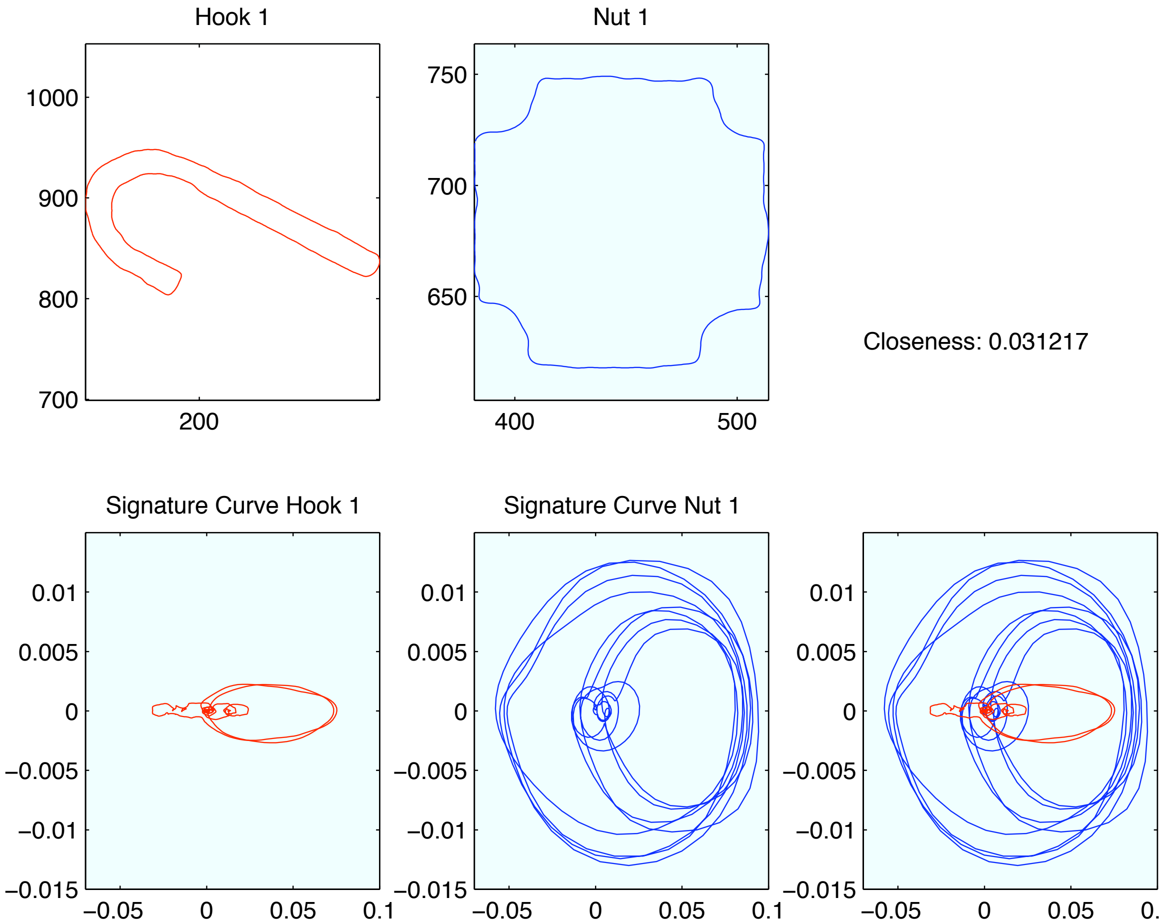


“Industrial Mathematics”

$\Longrightarrow$ Steve Haker



Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
600
550
500
450
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Advantages of the Signature Curve

- Purely local — no ambiguities
- Symmetries and approximate symmetries
- Extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Signatures of Binary Forms

$\Longrightarrow$ Irina Kogan

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0 \;$ and $\; (J'(x), K'(x)) \neq 0.$

Theorem.

Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\, H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & U = cH^2 \\ 4n - 8 & \text{otherwise} \end{cases}$$

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

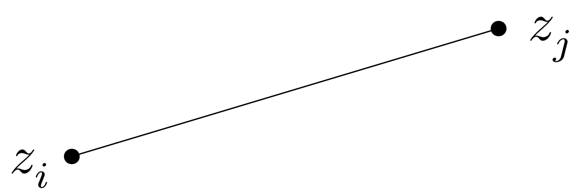
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

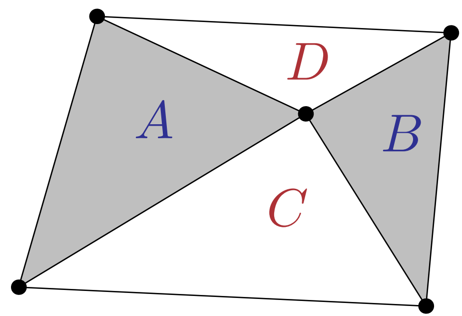
$$d(z_i, z_j) = \| z_i - z_j \|$$



Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

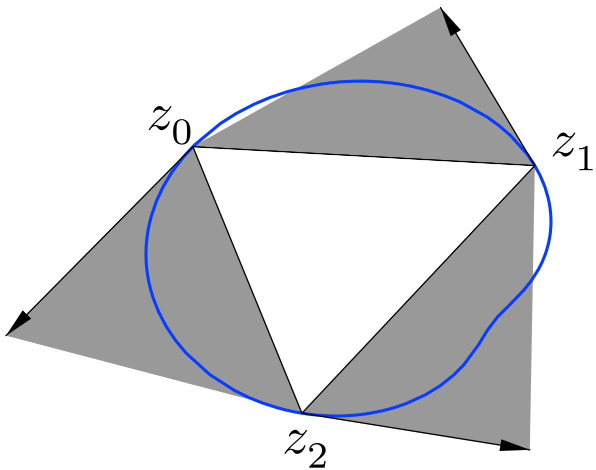
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$

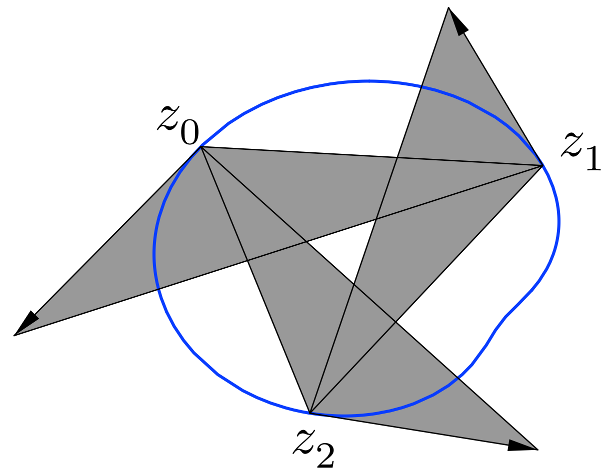


- Three–point projective joint differential invariant

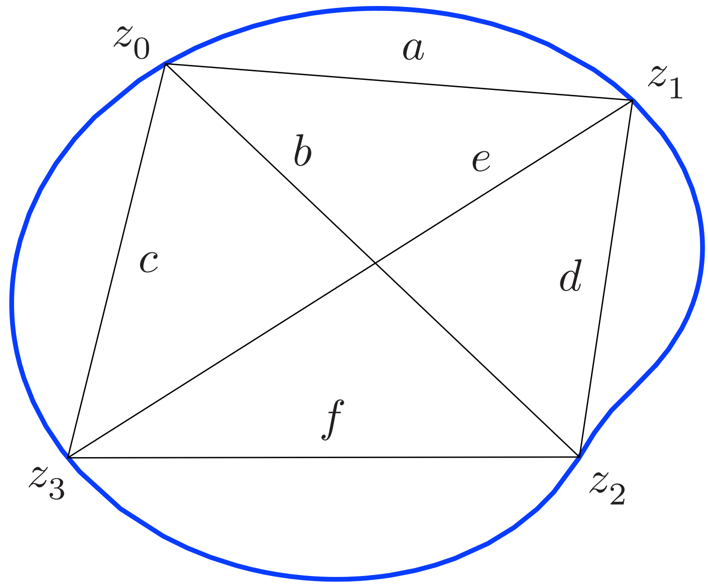
  — tangent triangle ratio:

$$\frac{[\,0\;2\;\dot{0}\,]\,[\,0\;1\;\dot{1}\,]\,[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\,[\,1\;2\;\dot{1}\,]\,[\,0\;2\;\dot{2}\,]}$$





Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$$\Longrightarrow \quad \text{six functions of four variables}$$

Syzygies:

$$\Phi_1(a,b,c,d,e,f) = 0 \qquad \Phi_2(a,b,c,d,e,f) = 0$$

Universal Cayley–Menger syzygy $\Longleftrightarrow$ $\mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2\,b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2\,c^2 \end{vmatrix} = 0$$

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.
- Multi–space (blow-up/Hilbert scheme?).

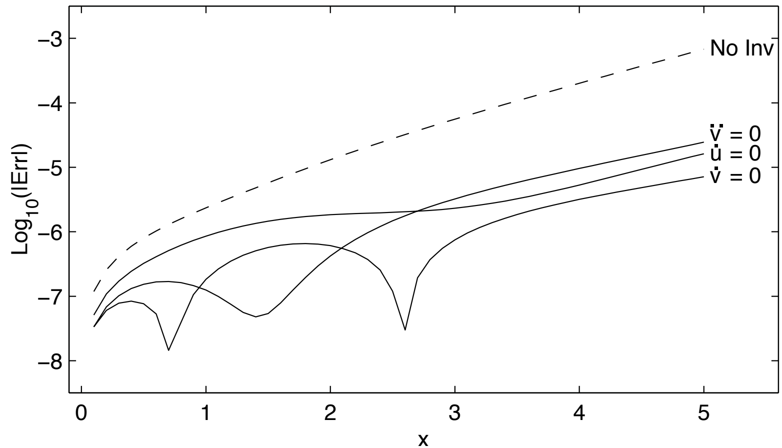
$\Longrightarrow$ Structure-preserving algorithms

Invariantization of Numerical Schemes

$\Longrightarrow$ Pilwon Kim

Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.

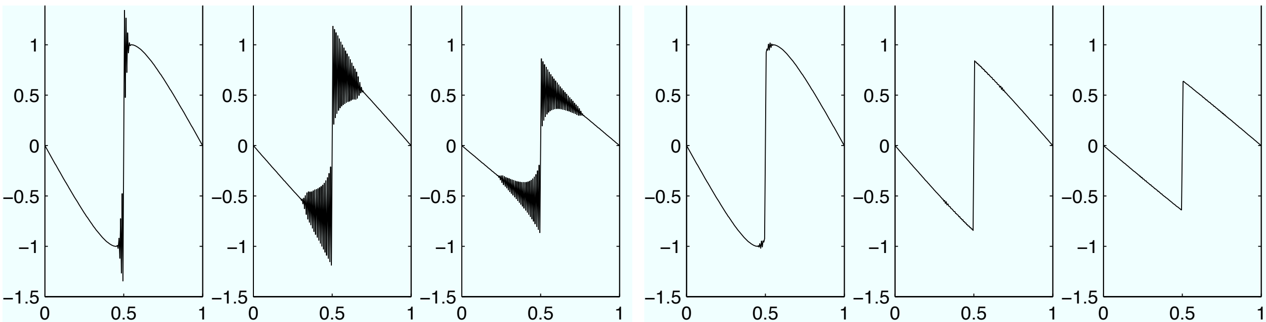


Invariant Runge–Kutta schemes

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$



Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\, \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\, dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[\,u\,] = \int ds = \int \sqrt{1 + u_x^2}\; dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[\,u\,] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[\,u\,] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$H^i(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

From the Invariant Variational Complex

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta)$$

$$\Longrightarrow \quad \vartheta \text{ — invariant contact form (variation)}$$

Invariant variation of curvature

$$\mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\,(ds) = \mathcal{B}(\vartheta) \wedge \, ds$$

Invariant variation of arc length:

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^* H^i(P) = (\mathcal{D}^2 + \kappa^2)\;\mathcal{E}(P) + \kappa\, H^i(P).$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\ \mathcal{E}(P) + \kappa\, H^i(P) = 0}$$

The Elastica: $\qquad \mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds \qquad P = \tfrac{1}{2}\,\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad\qquad H^i(P) = -\,P = -\,\tfrac{1}{2}\,\kappa^2$$

$$\boxed{\begin{aligned} \mathbf{E}(L) &= (\mathcal{D}^2 + \kappa^2)\ \kappa + \kappa\,(\,-\,\tfrac{1}{2}\,\kappa^2\,) \\ &= \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0 \end{aligned}}$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

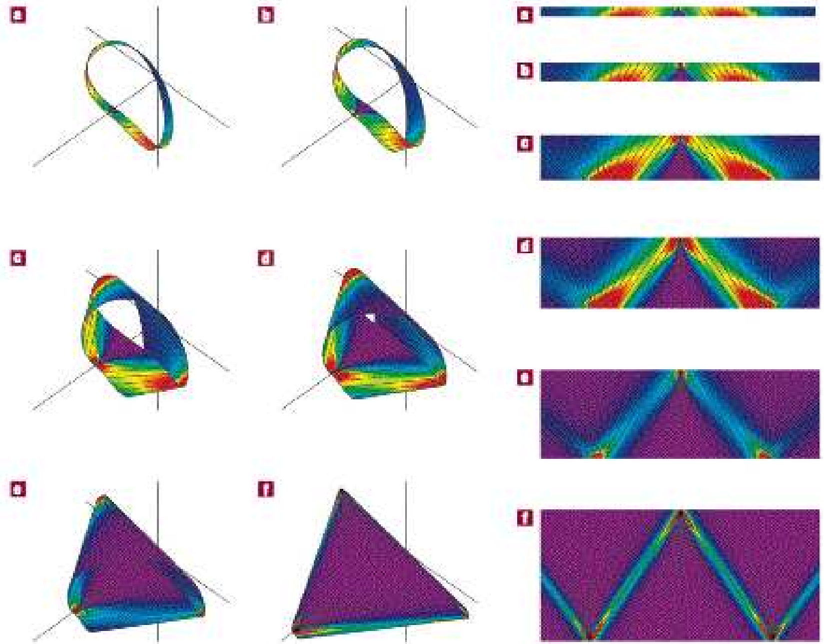


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

Evolution of Invariants and Signatures

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"
- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
  $$C_t = \mathbf{n}_{\text{equi-affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant arc length variation

$$d_{\mathcal{V}}\,(ds) = \mathcal{B}(\vartheta) \wedge ds$$

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A}_\kappa + \kappa\,\kappa_s$ — invariant variation of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

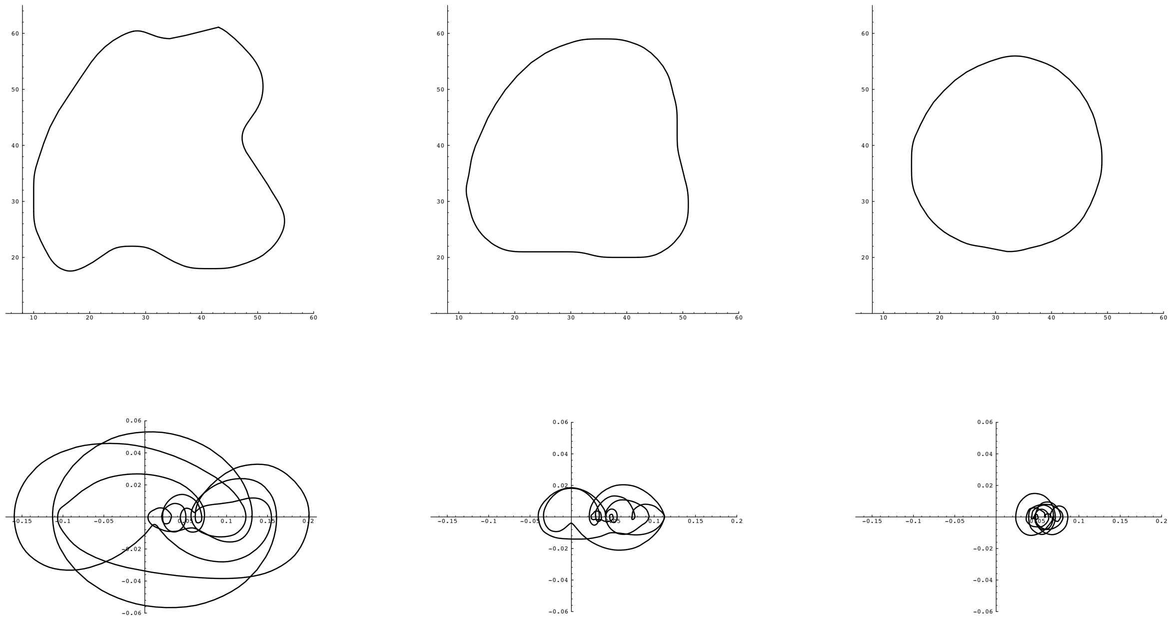
$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\,H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \tag{$*$}$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, Anco, and many more ...

Intrinsic Evolution of Differential Invariants

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Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = (\mathcal{D}^2 + \kappa^2)\, \vartheta, \qquad d_{\mathcal{V}}\, \varpi = -\kappa\, \vartheta \wedge \varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2 \kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta), \qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2, \qquad \mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa,$$

$$\begin{aligned} \mathcal{R} &= \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa \end{aligned}$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

Recursion operator:

$$\widehat{\mathcal{R}} = \mathcal{R}\cdot(\mathcal{D}^2 + \tfrac{1}{3}\,\kappa + \tfrac{1}{3}\,\kappa_s\mathcal{D}^{-1}).$$

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)