The Lagrangian Grassmannian, hyperbolic PDE, and G_2

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Workshop on Moving Frames in Geometry

Outline

Main theme:

Use surface theory in $\mathrm{LG}(2,4)$ (mod $CSp(4,\mathbb{R})$)
to study the geometry of PDE
$F(x,y,z,z_x,z_y,z_{xx},z_{xy},z_{yy})=0$

Outline:

1. A classification of (non-MA) hyperbolic PDE
2. Maximally symmetric "generic" hyperbolic PDE and G_2

$$\left(\text{e.g. } \frac{(3z_{xx}-6z_{xy}z_{yy}+2(z_{yy})^3)^2}{(2z_{xy}-(z_{yy})^2)^3}=c\right)$$

Motivation

- Non-MA hyperbolic PDE arise in hydrodynamic reduction of hyperbolic PDE in 3 indep vars (Smith, 2010)
- LG perspective on PDE in recent literature:
  1. Yamaguchi (1982)
  2. Ferapontov et al. (2009)
  3. Smith (2010)
  4. Doubrov–Ferapontov (2010)
  5. Alexeevsky et al. (2010)

What is a PDE? (Classical)

Definition

A PDE $F = 0$ is a hypersurface $\Sigma^7 \subset J^2(\mathbb{R}^2, \mathbb{R})$, transverse to $\pi^2_1 : J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$.

$$\begin{array}{rcll} \Sigma = F^{-1}(0) \subset & J^2(\mathbb{R}^2, \mathbb{R}) & : (x, y, z, p, q, r, s, t) \\ & \downarrow \pi^2_1 & \\ & J^1(\mathbb{R}^2, \mathbb{R}) & : (x, y, z, p, q) \end{array}$$

The jet spaces come equipped with contact systems:

1. J^1: $\sigma = dz - pdx - qdy$.
2. J^2: σ and $\sigma^1 = dp - rdx - sdy$, $\sigma^2 = dq - sdx - tdy$.

GOAL: Classify PDE up to (local) contact transformations.

What is a PDE? (Yamaguchi, 1982)

J : contact 5-mfld, i.e. $\exists$ corank 1 distribution $C = \{\sigma = 0\} \subset TJ$ s.t. $\eta = d\sigma$ on C is nondegenerate.

Darboux thm: $(J, C) \simeq_{loc} J^1(\mathbb{R}^2, \mathbb{R})$.

Definition

Given $(\mathbb{R}^4, \eta)$ symplectic, $\mathrm{LG}(2,4) :=$ isotropic 2-planes in $\mathbb{R}^4$.

Lagrange–Grassmann bundle $L(J) \xrightarrow{\pi} J$:

$$L(J) = \bigcup_{\xi \in J} \mathrm{LG}(C_\xi, [\eta]), \qquad \widetilde{C}_{\tilde{\xi}} = \pi_*^{-1}(\tilde{\xi}), \quad \tilde{\xi} \in L(J)|_\xi \subset C_\xi.$$

We have: $(L(J), \widetilde{C}) \simeq_{loc} J^2(\mathbb{R}^2, \mathbb{R})$.

Definition

A PDE is hypersurface in $L(J)$ transverse to $L(J) \xrightarrow{\pi} J$.

Locally speaking...

On J, have $\sigma = dz - pdx - qdy$, and

$$C = \{\sigma = 0\} = span\{\partial_x + p\partial_z,\, \partial_y + q\partial_z,\, \partial_p,\, \partial_q\},$$

and

$$\eta = d\sigma = dx \wedge dp + dy \wedge dq \sim \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \quad \text{on} \quad C.$$

Then at $\xi = (x, y, z, p, q)$,

$$(r, s, t) \leftrightarrow span\{\partial_x + p\partial_z + r\partial_p + s\partial_q, \partial_y + q\partial_z + s\partial_p + t\partial_q\}.$$

Contact transformations

- ϕ contact on $J \Leftrightarrow \phi_* C = C$. In fact, $\phi_* : (C, [\eta]) \to (C, [\eta])$ is conformal symplectomorphism.

  $$\boxed{\text{Prolongation to } L(J) := \phi_* = \text{induced map of } LG\text{'s.}}$$

- Backlünd thm:

  Φ contact on $L(J) \Rightarrow \boxed{\Phi = \phi_*}$ for ϕ contact on J.

Symplectic invariants yield contact invariants

IDEA: Do a fibrewise study of PDE.

i.e. Given $F(x, y, z, p, q, r, s, t) = 0$, freeze any $\xi = (x, y, z, p, q)$ and study the surface $F(r, s, t; \xi) = 0$ in $\mathrm{LG}(C_\xi) \cong \mathrm{LG}(2, 4)$.

Theorem (2010)

Any $CSp(4, \mathbb{R})$ differential invariant for surfaces in $\mathrm{LG}(2, 4)$ induces a contact invariant for PDE.

Generalizes to n-indep. vars. and to systems. (Only 1 dep. var.)

NOTE: This study only takes into account "vertical derivatives". e.g. Cannot distinguish btw $z_{xy} = 0$ or any hyperbolic MA PDE.

What's the point?: New invariants for non-MA PDE.

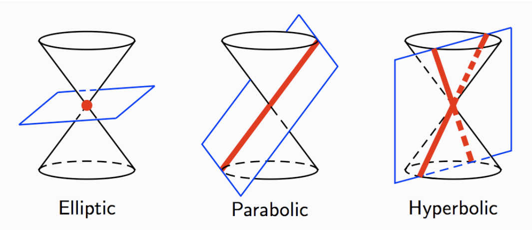
Elliptic, parabolic, hyperbolic PDE

$Sp(4,\mathbb{R})$ is SPECIAL: $\boxed{Sp(4,\mathbb{R}) \cong Spin(2,3)}$

Have a $CSp(4,\mathbb{R})$-invariant (Lorentzian) conformal structure $[\mu]$, so a cone $\mathcal{C} = \{\mu = 0\}$ in each tangent space of $\mathrm{LG}(2,4)$.

Classical description: Relative invariant $\Delta = F_r F_t - \frac{1}{4}(F_s)^2$.
Ell: $\Delta > 0$, par: $\Delta = 0$, hyp: $\Delta < 0$ (evaluated on $F = 0$).

LG perspective: Let $M^2 \subset \mathrm{LG}(2,4)$. $TM \cap \mathcal{C}$ looks like:



Projective realization and "spheres"

Plücker embedding: $\mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}(\bigwedge^2 \mathbb{R}^4)$. This restricts to $\mathrm{LG}(2,4) \hookrightarrow \mathbb{P}V = \mathbb{RP}^4$, where $V = \{z \in \bigwedge^2 \mathbb{R}^4 : \eta(z) = 0\}$.

On V, have sig. $(2,3)$ scalar product: $\langle \cdot, \cdot \rangle = \eta \wedge \eta$, and

$$\mathrm{LG}(2,4) = \mathcal{Q} = \{[z] \in \mathbb{P}V : \langle z, z \rangle = 0\}.$$

Definition

For any $[z] \in \mathbb{P}V$, we refer to $\mathcal{S}_{[z]} = \mathbb{P}(z^\perp) \cap \mathcal{Q}$ as a *"sphere"*.

i.e. if $[w] \in \mathcal{Q}$, we have $[w] \in \mathcal{S}_{[z]}$ iff $\langle w, z \rangle = 0$.

Thus, orthogonality $\leftrightarrow$ incidence!

Locally speaking...

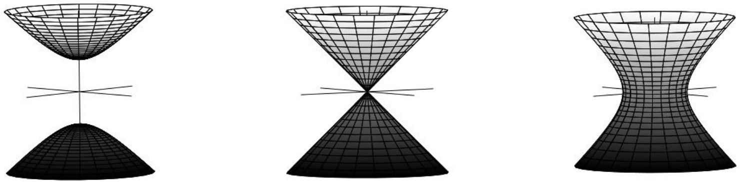
Take $\eta = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ wrt $\{e_1, \ldots, e_4\}$. Let $o = span\{e_1, e_2\}$. Then

1. $\mathrm{LG}(2,4) = CSp(4,\mathbb{R})/P$, where $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.
2. Nbd. of o is $\begin{pmatrix} I_2 & 0 \\ X & I_2 \end{pmatrix} /P$, where $X = \begin{pmatrix} r & s \\ s & t \end{pmatrix}$
   $\leftrightarrow span\{e_1 + re_3 + se_4,\ e_2 + se_3 + te_4\}$.
3. Conformal structure: $[\mu] = [drdt - ds^2]$.
4. $(e_1 + re_3 + se_4) \wedge (e_2 + se_3 + te_4)$
   $= e_1 \wedge e_2 + re_3 \wedge e_2 + s(e_1 \wedge e_3 - e_2 \wedge e_4) + te_1 \wedge e_4 + (rt - s^2)e_3 \wedge e_4$

   $\boxed{(r,s,t) \leftrightarrow [1, r, s, t, rt - s^2] \in \mathcal{Q}}$, $\boxed{\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}}$
5. $\mathcal{S}_{[z]}: \quad 0 = \langle w, z \rangle = -z_0(rt - s^2) + z_3 r - 2z_2 s + z_1 t - z_4$.
   Fibrewise, this is exactly the Monge–Ampère PDE: it's a sphere.

Invariance of the Monge–Ampère PDE

There are 3 types of spheres $\mathcal{S}_{[z]}$ according to sign of $\langle z, z \rangle$:



Theorem (Classical)

The class of ell. / par. / hyp. MA PDE are contact invariant.

New proof: "sphere", ell., par., hyp. are all $CSp(4,\mathbb{R})$ inv. notions.

Moving frames – adaptations

GOAL: $CSp(4,\mathbb{R})$-inv. study of hyperbolic $M^2 \subset \mathcal{Q}^3 \subset \mathbb{P}V \cong \mathbb{RP}^4$.

NOTE: No intrinsic geometry. (Any surface is conformally flat.)

Use moving frames!

Geometric interpretation:
A frame $\mathrm{v} = (\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)$ of V is a 5-tuple of spheres.

Projective moving frame adaptations:

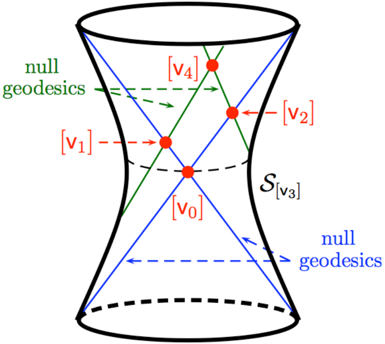
0. (a) $[\mathrm{v}_0] \in M$
   (b) $T_{\mathrm{v}_0}\widehat{\mathcal{Q}} = \mathrm{v}_0^\perp = span\{\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$. ($\widehat{\mathcal{Q}} = cone(\mathcal{Q})$)
1. (a) $T_{\mathrm{v}_0}\widehat{M} = span\{\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2\}$. ($\widehat{M} = cone(M)$)
   (b) Hyperbolic: Require $\overline{\mathrm{v}_1}, \overline{\mathrm{v}_2}$ to be null.
2. $\mathcal{S}_{[\mathrm{v}_3]}$ = central tangent sphere
3. If $M \neq$ sphere, $\exists$ normalizing cones $\mathcal{S}_{[\mathrm{v}_1]}, \mathcal{S}_{[\mathrm{v}_2]}$. Finally, $[\mathrm{v}_4] = \mathcal{S}_{[\mathrm{v}_1]} \cap \mathcal{S}_{[\mathrm{v}_2]} \cap \mathcal{S}_{[\mathrm{v}_3]}$ = conjugate point is determined.

Moving frames – geometric picture

For hyp. M, use hyp. frames v:

$$\langle v_i, v_j \rangle = \left(\begin{array}{c|cc|c|c} 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right)$$

Recall: orthogonality $\leftrightarrow$ incidence!

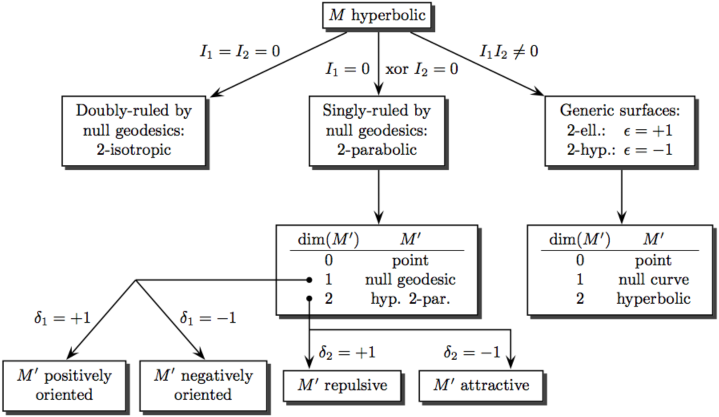


Definition

The conjugate manifold M' is the image of $M \to \mathcal{Q}$, $p \mapsto [v_4|_p]$.
Given PDE Σ, can fibrewise construct the conjugate PDE Σ'.

NOTE: Conjugation is not an involution!

Classification of hyperbolic surfaces / PDE



e.g. (i) $s = \frac{1}{2}t^2$: SR, M' pt; (ii) $3rt^3 + 1 = 0$ or $\frac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} = c$: gen., M' pt; (iii) $r = e^t$: gen., M' surface; (iv) $rt = -1$: gen. (Dupin cyclide), $M' = \{rt = -9\}$.

Maximally symmetric generic hyperbolic PDE

Definition

A hyperbolic PDE is of generic type if $I_1 I_2 \neq 0$, i.e. fibrewise, $\nexists$ null geodesics.

Theorem (Vranceanu 1937, T. 2008)

1. *Any gen. hyp. PDE has $\leq$ 9-dim contact sym [sharp].*
2. *All max. sym. models are given by*

A: $3rt^3 + 1 = 0$

B: $\dfrac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} = c$, *where* $c < -4$ *or* $c \geq 0$ $(*)$

$(*)$: *if* $c = 0$, *need* $s > \frac{t^2}{2}$ *for hyperbolicity.*

Degenerations to Cartan's G_2-models

Let $G = G_2$ (non-cpt). Relations to Cartan's 5-vars paper (1910):

1. $\dfrac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} = c$ has contact sym. alg. $\cong$ $\boxed{\mathfrak{p}_1 \subset \mathfrak{g}}$
2. $c = 0$: type-changing $3r - 6st + 2t^3 = 0$. Parabolic locus is Cartan's involutive system:

$$\boxed{r = \frac{t^3}{3}, \quad s = \frac{t^2}{2}}$$

3. $c = -4$: Cartan's parabolic Goursat model:

$$\boxed{9r^2 - 36rst + 12rt^3 - 12s^2t^2 + 32s^3 = 0}$$

Preview: The global picture

FACT: $J = G/P_2$ is a contact 5-mfld.

The G-action prolongs to $L(J) \to J$. Orbit decomposition:

$$L(J) = \mathcal{O}_8 \cup \mathcal{O}_7 \cup \mathcal{O}_6,$$

where

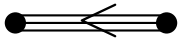
- $\mathcal{O}_8$ = open orbit;
- $\mathcal{O}_7$ = parabolic Goursat model;
- $\mathcal{O}_6$ = involutive system.

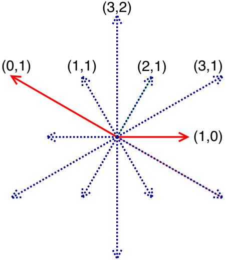
Theorem (2011)

The open orbit $\mathcal{O}_8 \subset L(J)$ is globally foliated by $\widetilde{P_1}$-orbits, all 7-dim. Moreover, every Type B max. sym. generic hyp. PDE occurs as a leaf in this foliation. (Note: $\exists$ other leaves.)

The parabolic subalgebra $\mathfrak{p}_2$

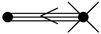
$\mathfrak{g}$:

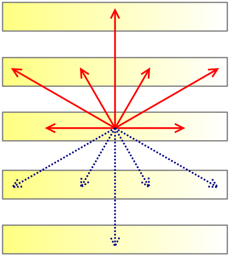




$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

$\mathfrak{p}_2$:





$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{p}_2 = \mathfrak{g}_{\geq 0}}$$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

Some $\mathfrak{sl}_2$-representation theory

For orbit decomp. of $L(J)$, look at fibre over $o \in J = G/P$.

1. $T_o(G/P) = \mathfrak{g}/\mathfrak{p} \supset \mathfrak{g}_{-1}/\mathfrak{p} = C_o$ (P-invariant).
2. Trivial $\mathfrak{g}_+$-action on C_o; reduce to $\mathfrak{g}_0$-action, where $\mathfrak{g}_0 = \mathfrak{gl}_2$.
3. GOAL: Understand GL_2-orbits on $\mathrm{LG}(C_o) = \mathcal{Q} \subset \mathbb{P}(\bigwedge^2_0 C_o)$.

As $\mathfrak{sl}_2$-reps,

$$\boxed{C_o = \Gamma_3 = S^3\mathbb{R}^2} \quad \text{and} \quad \boxed{\textstyle\bigwedge^2_0 C_o = \Gamma_4 = S^4\mathbb{R}^2}.$$

Clebsch–Gordan ($\mathfrak{sl}_2$-inv.) pairings give:

1. symplectic form η on Γ_3 (so, $\mathfrak{sl}_2 \to \mathfrak{sp}_4$)
2. sig. $(2,3)$ scalar product $\langle \cdot, \cdot \rangle$ on Γ_4 (so, $\mathfrak{sl}_2 \to \mathfrak{so}(2,3)$)

GL_2-orbits in $\mathcal{Q} \subset \mathbb{P}(\Gamma_4)$

On $\Gamma_4 = S^4(\mathbb{R}^2)$:

- $\langle f, f \rangle = 2 f_{xxxx} f_{yyyy} - 8 f_{xxxy} f_{yyyx} + 6 f_{xxyy} f_{yyxx}$.

On $\mathcal{Q} = \{[f] : \langle f, f \rangle = 0\} \subset \mathbb{P}(\Gamma_4)$, there are three GL_2-orbits:

GL_2-orbit	Description	Representative	G-orbit
$\mathcal{S}_1$	$v_4(\mathbb{P}^1)$	$[x^4]$	$\mathcal{O}_6$
$\mathcal{S}_2$	$\tau(\mathcal{S}_1) \backslash \mathcal{S}_1$	$[x^3 y]$	$\mathcal{O}_7$
$\mathcal{S}_3$	$\mathcal{Q} \backslash \tau(\mathcal{S}_1)$	$[xy(x^2 - \sqrt{3}xy + y^2)]$	$\mathcal{O}_8$

Here,

- $\mathcal{S}_1$ = rational normal quartic $= \{[a^4] : [a] \in \mathbb{P}^1\}$
- $\tau(\mathcal{S}_1)$ = tangential variety $= \{[a^3 b] : [a], [b] \in \mathbb{P}^1\}$

Coordinate description of GL_2-orbits

The induced $\mathfrak{sl}_2$-action in affine coords (r, s, t) on $\mathrm{LG}(C_o)$:

$$\begin{array}{ll} \mathrm{H}: & -3r\partial_r \quad -2s\partial_s \quad -t\partial_t \\ \mathrm{X}: & 4s^2\partial_r + (4st - 3r)\partial_s + (4t^2 - 6s)\partial_t \\ \mathrm{Y}: & -2s\partial_r \quad -t\partial_s \quad -\partial_t \end{array}$$

The GL_2-action has orbits:

1. $\mathcal{S}_1$: locally, $r = \frac{t^3}{3}, s = \frac{t^2}{2}$.

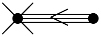
$$\mathbf{y} = (1, r, s, t, rt - s^2) = \left(1, \frac{t^3}{3}, \frac{t^2}{2}, t, \frac{t^4}{12}\right).$$

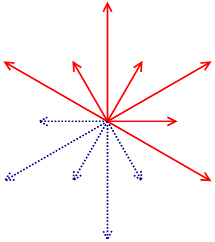
2. $\mathcal{S}_2$: locally, $9r^2 - 36rst + 12rt^3 - 12s^2t^2 + 32s^3 = 0$.

$$\mathbf{x} = (1, r, s, t, rt - s^2) = \mathbf{y} + u\mathbf{y}' \quad \Rightarrow \quad \frac{(3r - 6st + 2t^3)^2}{(2s - t^2)^3} = -4.$$

The parabolic subalgebra $\mathfrak{p}_1$

$\mathfrak{p}_1$:

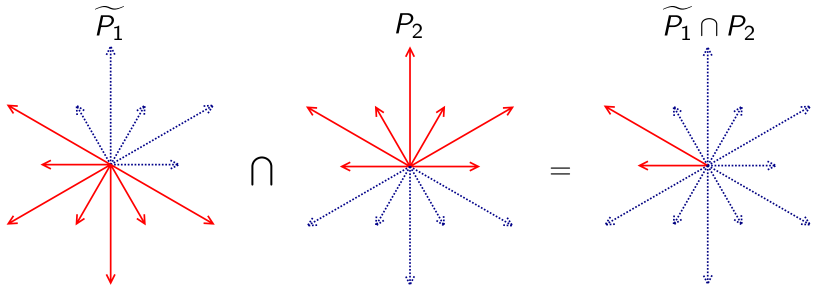




$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \underbrace{\oplus\, \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3}_{\mathfrak{p}_1 = \mathfrak{g}_{\geq 0}}$$

Flip P_1!

The *relative position* of P_1 wrt P_2 matters. Take $\boxed{\widetilde{P_1} = P_1^{op}}$.



$\widetilde{P_1} \cap P_2 =$ subgrp of $\widetilde{P_1}$ fixing $o \in J = G/P_2$:

- long root & grading elt act trivially on $\mathcal{Q} \cong \mathrm{LG}(C_o)$.
- has 2-dim orbits on $\mathcal{S}_3 \subset \mathcal{Q}$,
- locally, $\frac{(3r-6st+2t^3)^2}{(2s-t^2)^3}$ is a diff. inv. (i.e. preserved by H, Y)

The open orbit

Let $L \subset GL_2$ be the lower triangular 2×2 matrices.

Theorem

$\mathcal{S}_3 \subset \mathcal{Q}$ is globally foliated by L-orbits

- $\mathcal{T}_c$, $c \neq -4$:
  - *gen. hyp:* $c < -4$ *or* $c > 0$*; for* $c = 0$*, have* $\mathcal{T}_0^-$
  - *(gen.?) ell:* $0 < c < 4$*; for* $c = 0$*, have* $\mathcal{T}_0^+$
- $\mathcal{T}_\infty$ *: singly-ruled hyperbolic*
- $\mathcal{N}$ *: parabolic*

Using the $\widetilde{P_1}$*-action,* $\exists$ *corresponding foliation of* $\mathcal{O}_8 \subset L(J)$.

Eqns in local coords:

- $\mathcal{T}_c$: $\frac{(3r-6st+2t^3)^2}{(2s-t^2)^3} = c$.
- $\mathcal{T}_\infty$: $s = \frac{t^2}{2}$.
- $\mathcal{N}$: $rt - s^2 = 0$ (different chart).

Open questions

1. How to get PDE structure eqns adapted to moving frame adaptations in a fibre?
2. Is the conjugate PDE useful / interesting?
3. Submanifold theory in $LG(n, 2n)$ for $n \geq 3$? Geometrically interesting classes?