Geometry of curves in parabolic homogeneous spaces

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I Statement of the problem

Natural distributions on parabolic homogeneous spaces

G be a semisimple group
$\mathfrak{g}$ be its Lie algebra

P be a parabolic subgroup of G
$\mathfrak{p}$ be the Lie algebra of P

Then $\mathfrak{g}$ is equipped with the natural structure of graded Lie algebra:

$$\mathfrak{g} = \bigoplus_{i=-\mu}^{\mu} \mathfrak{g}_i \quad \text{s.t.}$$

$$\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i .$$

Natural distributions on parabolic homogeneous spaces

Consider the corresponding homogeneous space $M = G/P$

Let $o = eP$ be the coset of identity.
Then $T_o M \cong \mathfrak{g}/\mathfrak{p}$

Let D be the G-invariant distribution equal to $\mathfrak{g}_{-1} \bmod \mathfrak{p}$ at o - natural G-invariant distribution on G/P

We are interested in the problem of equivalence of integral curves of the natural distribution D on G/P with respect to the action of G (both for unparametrized and parametrized curves)

Motivation - geometry of curves of flags

Example 1

Fix integers $0 < k_1 \le k_2 \le \dots \le k_\mu < n$

Let $F_{k_1,\dots,k_\mu}(n)$ be the manifold of flags in $\mathbb{R}^n$

$$\Lambda_{-1} \subset \Lambda_{-2} \subset \dots \subset \Lambda_{-\mu} \subset \mathbb{R}^n$$ such that

$$\dim \Lambda_{-i} = k_i \qquad 1 \le i \le \mu$$

$$F_{k_1,\dots,k_\mu}(n) = SL(n,\mathbb{R}) / \underbrace{P}_{\text{a stationary subgroup of one of such flags}}$$

$\Lambda(t) = \{\Lambda_{-1}(t) \subset \dots \subset \Lambda_{-\mu}(t)\}$ is an integral curve of the natural distribution on $F_{k_1,\dots,k_\mu}(W)$ if and only if

$$\Lambda_i'(t) \subset \Lambda_{i-1}(t)$$

or equivalently $\Lambda_i'(t) \in \mathrm{Hom}\left(\Lambda_i(t), \Lambda_{i-1}(t) / \Lambda_i\right.$

compatibility w.r.t. differentiation.

Particular case - non-degenerate curves in the projective space $\mathbb{R}P^k$.

Non-degeneracy means that a curve does not lie in any proper linear subspace of $\mathbb{R}P^k$

Taking the osculating flag

$$\gamma \subset \gamma' \subset \gamma'' \dots \subset \gamma^{(k-1)}$$

one gets a curve of complete flags in $\mathbb{R}^{k+1}$ and it is an integral curve of the natural distibution in $SL(k+1,\mathbb{R})/\underbrace{B}_{\text{Borel subgroup}}$

Example 2 Curves of symplectic flags

Assume in addition that n is even and $\mathbb{R}^n$ is endowed with a symplectic structure σ.

A flag f is called symplectic if the following two conditions hold:

(1) any subspace in the flag f is either isotropic or coisotropic w.r.t. σ

(2) a subspace belongs to the flag f together with its skew-symmetric complement w.r.t. σ

We are interested in geometry of curves of symplectic flags compatible w.r.t. differention – naturally appear in intinsic description of Jacobi eq. along extremals.

Algebraic version of Cartan's method of moving frames or Tanaka's type prolongation for curves in G/P

The first natural invariant of an integral curve of D is a "type" of its tangent line.

Let $\mathbb{P}D$ be the projectivization of D. The action of G on G/P is naturally lifted to $\mathbb{P}D$ and it is, in general, no longer transitive. The orbits of this action are in one-to-one correspondence with the orbits of the adjoint action of P on the projective space $\mathbb{P}\mathfrak{g}_{-1}$ or, equivalently, of the adjoint action of G_0 on $\mathbb{P}\mathfrak{g}^{-1}$, where G_0 is the subgroup of G with Lie algebra $\mathfrak{g}_0$.

The symbol of an integral curve γ of D at a point

Each integral curve γ of D in G/P is naturally lifted to the curve in PD

The symbol of the curve γ at a point x is the orbit of the aforementioned action of G_0 on $\mathbb{P}\mathfrak{g}_{-1}$ corresponding to the tangent line $T_x\gamma$ to γ at x.

Thm (E. Vinberg, 1976) The action of the group G_0 on $\mathbb{P}(\mathfrak{g}_{-1})$ has a finite number of orbits

$\Downarrow$

Any integral curve of D in a neighb. of a generic point has a constant symbol

It is natural to restrict ourselves to the case of integral curves with constant symbol.

Fix an element $m \in \mathbb{P}\,\mathfrak{g}_{-1}$ and consider all integral curves of D in G/P with the symbol at any point which is an orbit of m w.r.t. the action of G_0 on $\mathbb{P}\,\mathfrak{g}_{-1}$.

In this case we also say that a curve has constant type m

Our goal To any curve of type m to assign the canonical bundle of moving frames in a unified way

The flat curve of type $\mathfrak{m}$

Take the one-parametric subgroup of G generated by $\mathfrak{m}$ and let $\gamma_{\mathfrak{m}}$ be its image under the canonical projection to G/P.

A curve is called flat of type $\mathfrak{m}$ if it is G-equivalent to $\gamma_{\mathfrak{m}}$

Universal algebraic prolongation $u(\mathfrak{m})$ of $\mathfrak{m}$ is the largest graded subalgebra of $\mathfrak{g}$ s.t. its negative part is equal to $\mathfrak{m}$

Explicit construction of $u(\mathfrak{m})$

Set $u_i := 0$, $i < -1$ and $u_{-1} := \mathfrak{m}$

$u_0 = \{x \in \mathfrak{g}_0 : [\mathfrak{m}, x] \subset \mathfrak{m}\}$ - 0-prolongation

$u_0 = \{ x \in \mathfrak{g}_0 \mid [\mathfrak{m}, x] \subset \mathfrak{m} \}$ - 0-prolongation

$u_1 = \{ x \in \mathfrak{g}_1 \mid [\mathfrak{m}, x] \subset u_0 \}$ - 1st prolongation

$\vdots$

By induction

$u_i = \{ x \in \mathfrak{g}_i \mid [\mathfrak{m}, x] \in u_{i-1} \}$ - the ith prolongation

Then

$$u(\mathfrak{m}) = \bigoplus_{i \geq -1} u_i$$

Geometric interpretation of $u(\mathfrak{m})$

If $S(\mathfrak{m})$ $(\subset G)$ is a symmetry group of the flat curve $\gamma_{\mathfrak{m}}$ then $u(\mathfrak{m})$ is exactly the Lie algebra of $S(\mathfrak{m})$

As in Tanaka's theory for filtered structures on manifolds we want to imitate the construction of the canon. bundle of moving frames for any curve of type $\mathfrak{m}$ via the construction of such bundle for the flat curve $\gamma_{\mathfrak{m}}$.

Bundles of moving frames.

Let $\pi : G \to G/P$ be the canonical principle P-bundle

Let $\omega : TG \to \mathfrak{g}$ be the left invariant Maurer-Cartan form.

By a moving frame bundle over a curve γ we mean any subbundle E (not necessary principle) of the P-bundle $\pi^{-1}(\gamma) \to \gamma$ endowed with an Ehresmann connection C (i.e. with a rank 1 distribution C transversal to the fibers of E)

Some notions on filtered and graded spaces

If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a graded space, then V has a natural decreasing filtration $\{V^{(k)}\}$, where $V^{(k)} = \bigoplus_{i \geq k} V_i$

If L is a subspace of V then the filtration $\{V^{(k)}\}$ defines the filtration $\{L^{(k)}\}$ on L: $L^{(k)} = L \cap V^{(k)}$.

Let
$$\mathrm{gr}L = \bigoplus_{i \in \mathbb{Z}} L^{(i)}/L^{(i+1)},$$
where $L^{(i)}/L^{(i+1)}$ can be naturally identified with a subspace of V_i

$\mathrm{gr}L \subset V$

Normalization conditions

Let $\mathfrak{u}^+(\mathfrak{m}) = \bigoplus_{i\geq 0} \mathfrak{u}_i$ be a non-negative part of the universal algebraic prolongation $\mathfrak{u}(\mathfrak{m})$ of $\mathfrak{m}$

Let $S^+(\mathfrak{m}) = \underbrace{S(\mathfrak{m})}_{\text{the symmetry group of } \delta_{\mathfrak{m}}} \cap P$

A normalization condition is any graded subspace $N \subset \mathfrak{p}$ complementary to $\boxed{\mathfrak{u}^+(\mathfrak{m}) + [\mathfrak{m}, \mathfrak{p}]}$

In other words, $N = \bigoplus_{i\geq 0} N_i$, where N_i is a subspace of $\mathfrak{g}_i$ complementary to $\mathfrak{u}_i + [\mathfrak{m}, \mathfrak{g}_{i+1}]$

N is called $S^+(\mathfrak{m})$-invariant if it is invariant w.r.t. the adjoint action of $S^+(\mathfrak{m})$

Main theorem (B. Doubrov & I. Z.)

Fix a normalization condition N.

Then for any curve γ of type m there exists the unique frame bundle $E \subset G$ along γ with Ehresmann connection C such that $\dim E = \dim u(m)$ and for each point $q \in E$ the following 2 conditions hold:

(1) $\omega(C(q)) \subset m \oplus N$

(2) $\operatorname{gr} \omega(T_q E) = u(m)$

If in addition N is $S^+(m)$-invariant then E is a principal $S^+(m)$-bundle, C is a principal connection, and

$u^+(m) \subset \omega(T_q E) \subset u(m) \oplus N$

Moreover, $\omega|_E$ decomposes as a sum of the N- valued 1-form ω_N and the u-valued 1-form ω_u, where ω_u is a flat Cartan connection on γ and ω_N is a vertical $S^+(m)$-equivariant 1-form defining the fundamental set of differential invariants of γ (i.e. all differential invariants of γ can be obtained from the form ω_N and all its covariant derivatives defined by the Cartan connection ω_u

In particular, the curve γ is locally flat $\Longleftrightarrow$ ω_N vanishes identically.

General structure of the universal prolongation $\mathfrak{u}(\mathfrak{m})$

Fix $X \in \mathfrak{m}$ By Jacobson-Morozov thm one can complete X by elements H and Y of degree 0 and 1 respectively (in $\mathfrak{g}$) to the standard basis of the subalgebra $sl_2 \subset \mathfrak{g}$, i.e. s.t. $[H,X]=2X$, $[H,Y]=-2Y$, $[X,Y]=H$

Since $\mathfrak{u}(\mathfrak{m})$ is an algebra of infinitesimal symmetries of the flat curve $\gamma_{\mathfrak{m}}$ to any element of $\mathfrak{u}(\mathfrak{m})$ one can assign a vector field on the curve $\gamma_{\mathfrak{m}}$

Let $\mathfrak{n}_{ne}$ be a subspace of $\mathfrak{u}(\mathfrak{m})$ consisting of all elements for which the corresponding vector fields on $\gamma_{\mathfrak{m}}$ are identically equal to 0

$\mathfrak{n}_{ne}$ is called the non-effectiveness ideal of $\mathfrak{u}(\mathfrak{m})$

Then $\mathfrak{u}(\mathfrak{m})$ is a semidirect sum of $\underbrace{\mathfrak{sl}_2}_{\langle H,X,Y\rangle}$ and $\mathfrak{n}_{ne}$.

$\Downarrow$

If $S^+(\mathfrak{m})$-invariant normalization condition exists $\Rightarrow$ ω_u defines the canonical projective structure on a curve of type $\mathfrak{m}$

Some conditions for existence of $S^+(\mathfrak{m})$-invariant normalization condition and how to find it.

Assume that $\mathfrak{u}(\mathfrak{m})$ is reductive.
Then the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{u}(\mathfrak{m})$ is non-degenerate.
Let $\mathfrak{u}(\mathfrak{m})^\perp$ be the orthogonal complement of $\mathfrak{u}(\mathfrak{m})$ w.r.t. the Killing form
Then $N = \{ w \in \mathfrak{U}(\mathfrak{m})^\perp : [u, Y] = 0 \}$ is an $S^+(\mathfrak{m})$- invariant normalization condition.

<u>Example</u> $G = SL_{k+1}(\mathbb{R})/B$ - the variety of complete flags in $\mathbb{R}^{k+1}$ - geometry of non-degenerate curves in $\mathbb{R}P^k$

$\mathfrak{m} = \langle X \rangle$, where

$$X = \begin{pmatrix} 0 & \cdots & & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$ - a nilpotent Jordan block

The non-effectiveness ideal $\mathfrak{n}_{ne} = 0$ so $\mathfrak{u}(\mathfrak{m}) \cong \mathfrak{sl}_2(\mathbb{R})$

$\mathfrak{u}(\mathfrak{m})$ is the irreducible embedding of $\mathfrak{sl}_2(\mathbb{R})$ into $\mathfrak{sl}_{k+1}(\mathbb{R})$

A $\mathcal{S}^+(\mathfrak{m})$ - invariant normalization condition $N = \operatorname{span}\langle Y^i, i=2,..k\rangle$

$\omega_N(C) = \sum \underbrace{A_i}_{} Y^i$

Wilczynski invariants

In general, $S^+(m)$-invariant normalization may not exist.

<u>Example</u> $F_{2,4}(\mathbb{R}^5)$

$$t \to \underbrace{\Lambda_{-1}(t)}_{2\,\text{dim}} \subset \underbrace{\Lambda_{-2}(t)}_{4\,\text{dim}} \subset \mathbb{R}^5$$

$$\Lambda_{-1}'(t) = \Lambda_{-2}(t), \quad \Lambda_{-2}'(t) = \mathbb{R}^5$$

<u>Rem 1</u> $S^+(m)$-invariant normalization may exist even if $u(m)$ is not reductive

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<u>Rem 2</u> For parametrized curves there are two modifications:

1) a symbol is just an orbit of adjoint action of G_0 on $\mathfrak{g}_{-1}$
2) If $\delta \in \mathfrak{g}_{-1}$, then $u_{-1}(\delta) := \operatorname{span} \delta$, $u_0(\delta) := \{X \in \mathfrak{g}_0 \mid [X, \delta] = 0\}$

<u>Rem 3</u> A similar theory can be constructed for integral submanifold of the natural distribution D on G/P