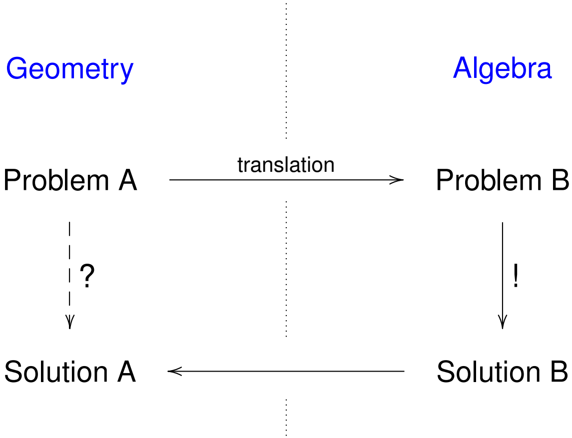
Separable Coordinates on the 3-Sphere

Konrad Schöbel

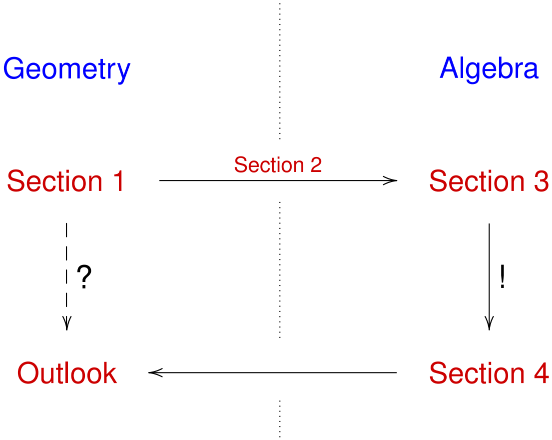
Friedrich-Schiller-Universität Jena

Workshop on Moving Frames in Geometry

Montreal, 13 – 17 June 2011


Geometry
Algebra
Problem A
translation
Problem B
?
!
Solution A
Solution B

Outline


Geometry
Algebra
Section 1
Section 2
Section 3
?
!
Outlook
Section 4

Outline

Outline

Separation of variables

General problem

Classify all coordinate systems in which a given partial differential equation is solvable by a separation of variables.

here: Hamilton-Jacobi equation

$$\tfrac{1}{2} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} + V = E$$

Integrable Killing tensors

(M, g) Riemannian manifold

Definition

- K Killing tensor $:\Leftrightarrow$

$$K_{\beta\gamma} = K_{\gamma\beta} \qquad \nabla_{(\alpha} K_{\beta\gamma)} = 0$$

- K integrable $:\Leftrightarrow$

  $\exists$ coordinates x_α in a neighbourhood of almost every point:

$$K^{\beta}{}_{\gamma}\,\partial_\beta = \lambda(x)\partial_\gamma$$

  i. e. coordinate vectors = eigen vectors of K

Integrable Killing tensors

(M, g) Riemannian manifold

Definition

- K Killing tensor $:\Leftrightarrow$

$$K_{\beta\gamma} = K_{\gamma\beta} \qquad \nabla_{(\alpha} K_{\beta\gamma)} = 0$$

- K integrable $:\Leftrightarrow$

  $\exists$ coordinates x_α in a neighbourhood of almost every point:

$$K^{\beta}{}_{\gamma}\,\partial_\beta = \lambda(x)\partial_\gamma$$

  i. e. coordinate vectors = eigen vectors of K

via integrable Killing tensors

Theorem (Stäckel, Eisenhart, Benenti)

Every orthogonal coordinate system in which the Hamilton-Jacobi equation separates is given by

1. *an integrable Killing tensor K*
2. *with pointwise simple eigen values*
3. *compatible with the potential:* $d(KdV) = 0$

First step

Determine integrable Killing tensors.

classical approach: Moving Frames

VIA INTEGRABLE KILLING TENSORS

THEOREM (STÄCKEL, EISENHART, BENENTI)

Every orthogonal coordinate system in which the Hamilton-Jacobi equation separates is given by

1. *an integrable Killing tensor K*
2. *with pointwise simple eigen values*
3. *compatible with the potential:* $d(KdV) = 0$

FIRST STEP

Determine integrable Killing tensors.

classical approach: Moving Frames

via integrable Killing tensors

Theorem (Stäckel, Eisenhart, Benenti)

Every orthogonal coordinate system in which the Hamilton-Jacobi equation separates is given by

1. *an integrable Killing tensor K*
2. *with pointwise simple eigen values*
3. *compatible with the potential:* $d(KdV) = 0$

First step

Determine integrable Killing tensors.

classical approach: Moving Frames

NIJENHUIS INTEGRABILITY

DEFINITION

Nijenhuis torsion:

$$N(X,Y) := K^2[X,Y] - K[KX,Y] - K[X,KY] + [KX,KY]$$

$$N^{\alpha}{}_{\beta\gamma} = K^{\alpha}{}_{\delta}\,\nabla_{[\gamma}K^{\delta}{}_{\beta]} + \nabla_{\delta}K^{\alpha}{}_{[\gamma}\,K^{\delta}{}_{\beta]}$$

THEOREM (TONOLO, SCHOUTEN, NIJENHUIS '51)

$$K \quad \textit{integrable} \quad \Leftrightarrow \quad \begin{cases} 0 = N^{\delta}{}_{[\beta\gamma}\,g_{\alpha]\delta} & \textit{(NI)} \\ 0 = N^{\delta}{}_{[\beta\gamma}\,K_{\alpha]\delta} & \textit{(NII)} \\ 0 = N^{\delta}{}_{[\beta\gamma}\,K_{\alpha]\varepsilon}\,K^{\varepsilon}{}_{\delta} & \textit{(NIII)} \end{cases}$$

PRECISE PROBLEM

Solve (NI)–(NIII) for Killing tensors on S^3.

NIJENHUIS INTEGRABILITY

DEFINITION

Nijenhuis torsion:

$$N(X,Y) := K^2[X,Y] - K[KX,Y] - K[X,KY] + [KX,KY]$$

$$N^{\alpha}{}_{\beta\gamma} = K^{\alpha}{}_{\delta}\,\nabla_{[\gamma}K^{\delta}{}_{\beta]} + \nabla_{\delta}K^{\alpha}{}_{[\gamma}\,K^{\delta}{}_{\beta]}$$

THEOREM (TONOLO, SCHOUTEN, NIJENHUIS '51)

$$K \quad \textit{integrable} \quad \Leftrightarrow \quad \begin{cases} 0 = N^{\delta}{}_{[\beta\gamma}\,g_{\alpha]\delta} & \textit{(NI)} \\ 0 = N^{\delta}{}_{[\beta\gamma}\,K_{\alpha]\delta} & \textit{(NII)} \\ 0 = N^{\delta}{}_{[\beta\gamma}\,K_{\alpha]\varepsilon}\,K^{\varepsilon}{}_{\delta} & \textit{(NIII)} \end{cases}$$

PRECISE PROBLEM

Solve (NI)–(NIII) for Killing tensors on S^3.

Benenti tensors

Theorem (Matveev-Topalov '98)

$$g' \sim g \qquad \Rightarrow \qquad K := \left(\frac{\det g}{\det g'}\right)^{\frac{2}{n+1}} g' \quad \textit{Killing tensor for } g$$

Example

$$f: \quad S^n \to S^n, \quad x \mapsto f(x) := \frac{Ax}{\|Ax\|} \qquad A \in GL(n+1) \qquad f^*g \sim g$$

Definition

K Benenti Killing tensor

Particular solution

Benenti Killing tensors are integrable.

Benenti tensors

Theorem (Matveev-Topalov '98)

$$g' \sim g \qquad \Rightarrow \qquad K := \left(\frac{\det g}{\det g'}\right)^{\frac{2}{n+1}} g' \quad \textit{Killing tensor for } g$$

Example

$$\begin{array}{rrcl} f: & S^n & \to & S^n \\ & x & \mapsto & f(x) := \frac{Ax}{\|Ax\|} \end{array} \qquad A \in GL(n+1) \qquad f^*g \sim g$$

Definition

K Benenti Killing tensor

Particular solution

Benenti Killing tensors are integrable.

Benenti tensors

Theorem (Matveev-Topalov '98)

$$g' \sim g \qquad \Rightarrow \qquad K := \left(\frac{\det g}{\det g'}\right)^{\frac{2}{n+1}} g' \quad \textit{Killing tensor for } g$$

Example

$$\begin{array}{rccc} f: & S^n & \to & S^n \\ & x & \mapsto & f(x) := \frac{Ax}{\|Ax\|} \end{array} \qquad A \in GL(n+1) \qquad f^* g \sim g$$

Definition

K Benenti Killing tensor

Particular solution

Benenti Killing tensors are integrable.

Benenti tensors

Theorem (Matveev-Topalov '98)

$$g' \sim g \qquad \Rightarrow \qquad K := \left(\frac{\det g}{\det g'}\right)^{\frac{2}{n+1}} g' \quad \textit{Killing tensor for } g$$

Example

$$\begin{array}{rrcl} f: & S^n & \to & S^n \\ & x & \mapsto & f(x) := \frac{Ax}{\|Ax\|} \end{array} \qquad A \in GL(n+1) \qquad f^*g \sim g$$

Definition

K Benenti Killing tensor

Particular solution

Benenti Killing tensors are integrable.

Outline

Algebraic curvature tensors

Definition

$R_{a_1 b_1 a_2 b_2} \in (V^*)^{\otimes 4}$ algebraic curvature tensor on V $:\Leftrightarrow$

1. antisymmetry:

$$R_{b_1 a_1 a_2 b_2} = -R_{a_1 b_1 a_2 b_2} = R_{a_1 b_1 b_2 a_2}$$

2. pair symmetry:

$$R_{a_2 b_2 a_1 b_1} = +R_{a_1 b_1 a_2 b_2}$$

3. Bianchi identity:

$$R_{a_1 b_1 a_2 b_2} + R_{a_1 a_2 b_2 b_1} + R_{a_1 b_2 b_1 a_2} = 0$$

algebraic curvature tensors $R_{a_1 b_1 a_2 b_2}$ $=$ $GL(V)$-irrep $\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline b_1 & b_2 \\ \hline \end{array}^{\,\star}$

Algebraic curvature tensors

Definition

$R_{a_1 b_1 a_2 b_2} \in (V^*)^{\otimes 4}$ algebraic curvature tensor on V $:\Leftrightarrow$

1. antisymmetry:

$$R_{b_1 a_1 a_2 b_2} = -R_{a_1 b_1 a_2 b_2} = R_{a_1 b_1 b_2 a_2}$$

2. pair symmetry:

$$R_{a_2 b_2 a_1 b_1} = +R_{a_1 b_1 a_2 b_2}$$

3. Bianchi identity:

$$R_{a_1 b_1 a_2 b_2} + R_{a_1 a_2 b_2 b_1} + R_{a_1 b_2 b_1 a_2} = 0$$

algebraic curvature tensors $R_{a_1 b_1 a_2 b_2}$ = $GL(V)$-irrep $\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline b_1 & b_2 \\ \hline \end{array}^{\star}$

Algebraic description of Killing tensors

on constant curvature manifolds

$$S^n \subset V \qquad\qquad Isom(S^n) = O(V) \subset GL(V)$$

Theorem (McLenaghan, Milson & Smirnov '04)

There is an isomorphism of $O(V)$-representations

$$\text{Killing tensors } K \text{ on } S^n \quad \longleftrightarrow \quad \text{algebraic curvature tensors } R \text{ on } V$$

$$K_x(v, w) \quad := \quad R_{a_1 b_1 a_2 b_2} x^{a_1} x^{a_2} v^{b_1} w^{b_2}$$

$$x \in S^n \qquad v, w \perp x$$

(and similarly for all constant curvature manifolds).

Algebraic description of Killing tensors
on constant curvature manifolds

$$S^n \subset V \qquad\qquad Isom(S^n) = O(V) \subset GL(V)$$

Theorem (McLenaghan, Milson & Smirnov '04)

There is an isomorphism of $O(V)$-representations

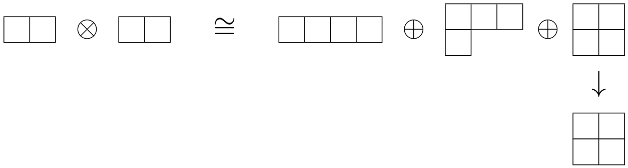
$$\text{Killing tensors } K \text{ on } S^n \quad \longleftrightarrow \quad \text{algebraic curvature tensors } R \text{ on } V$$

$$K_x(v, w) \quad := \quad R_{a_1 b_1 a_2 b_2} x^{a_1} x^{a_2} v^{b_1} w^{b_2}$$

$$x \in S^n \qquad v, w \perp x$$

(and similarly for all constant curvature manifolds).

The Kulkarni-Nomizu product



Definition

Kulkarni-Nomizu product $h \owedge k$ of symmetric tensors $h_{a_1 a_2}$ and $k_{b_1 b_2}$

$$(h \owedge k)_{a_1 b_1 a_2 b_2} = h_{a_1 a_2} k_{b_1 b_2} - h_{a_1 b_2} k_{b_1 a_2} - h_{b_1 a_2} k_{a_1 b_2} + h_{b_1 b_2} k_{a_1 a_2}$$

The metric & Benenti Killing tensors

as algebraic curvature tensors

Example

metric		algebraic curvature tensor
g	$\longleftrightarrow$	$\frac{1}{2} g \owedge g$

Example

Benenti Killing tensor		algebraic curvature tensor
K_A	$\longleftrightarrow$	$\frac{1}{2}(Ag) \owedge (Ag)$

where

$$A \in GL(V) \qquad (Ag)(v, w) := g(Av, Aw)$$

Benenti Killing tensors $=$ $GL(V)$-orbit of the metric

The metric & Benenti Killing tensors
as algebraic curvature tensors

Example

metric		algebraic curvature tensor
g	$\longleftrightarrow$	$\frac{1}{2} g \owedge g$

Example

Benenti Killing tensor		algebraic curvature tensor
K_A	$\longleftrightarrow$	$\frac{1}{2}(Ag) \owedge (Ag)$

where

$$A \in GL(V) \qquad (Ag)(v, w) := g(Av, Aw)$$

Benenti Killing tensors = $GL(V)$-orbit of the metric

THE METRIC & BENENTI KILLING TENSORS

AS ALGEBRAIC CURVATURE TENSORS

EXAMPLE

metric g $\longleftrightarrow$ algebraic curvature tensor $\frac{1}{2} g \owedge g$

EXAMPLE

Benenti Killing tensor K_A $\longleftrightarrow$ algebraic curvature tensor $\frac{1}{2}(Ag) \owedge (Ag)$

where

$$A \in GL(V) \qquad (Ag)(v, w) := g(Av, Aw)$$

Benenti Killing tensors = $GL(V)$-orbit of the metric

Outline

Algebraic integrability conditions

Theorem (–)

A Killing tensor on S^n is integrable $\Leftrightarrow$ the corresponding algebraic curvature tensor $R_{a_1 b_1 a_2 b_2}$ satisfies

$$\begin{array}{|c|c|c|}\hline a_2 & b_1 & d_1 \\ \hline b_2 \\ \cline{1-1} c_2 \\ \cline{1-1} d_2 \\ \cline{1-1} \end{array}^{\,\star} \; g_{ij} R^i{}_{b_1 a_2 b_2} R^j{}_{d_1 c_2 d_2} = 0$$

$$\begin{array}{|c|c|c|c|c|}\hline a_2 & a_1 & b_1 & c_1 & d_1 \\ \hline b_2 \\ \cline{1-1} c_2 \\ \cline{1-1} d_2 \\ \cline{1-1} \end{array}^{\,\star} \; g_{ij} g_{kl} R^i{}_{b_1 a_2 b_2} R^j{}_{a_1}{}^k{}_{c1} R^l{}_{d_1 c_2 d_2} = 0$$

(and similarly for all constant curvature manifolds).

ADVANTAGES

OF THE ALGEBRAIC APPROACH

1. simple algebraic equations instead of non-linear PDE system
2. third equation redundant!
3. valid for all (pseudo-)Riemannian constant curvature manifolds
4. Riemann tensors are intensively studied.
5. new insight into integrability from
   - representation theory
   - algebraic geometry
   - geometric invariant theory

ADVANTAGES

OF THE ALGEBRAIC APPROACH

1. simple algebraic equations instead of non-linear PDE system
2. third equation redundant!
3. valid for all (pseudo-)Riemannian constant curvature manifolds
4. Riemann tensors are intensively studied.
5. new insight into integrability from
    - representation theory
    - algebraic geometry
    - geometric invariant theory

ADVANTAGES

OF THE ALGEBRAIC APPROACH

1. simple algebraic equations instead of non-linear PDE system
2. third equation redundant!
3. valid for all (pseudo-)Riemannian constant curvature manifolds
4. Riemann tensors are intensively studied.
5. new insight into integrability from
   - representation theory
   - algebraic geometry
   - geometric invariant theory

ADVANTAGES

OF THE ALGEBRAIC APPROACH

1. simple algebraic equations instead of non-linear PDE system
2. third equation redundant!
3. valid for all (pseudo-)Riemannian constant curvature manifolds
4. Riemann tensors are intensively studied.
5. new insight into integrability from
   - representation theory
   - algebraic geometry
   - geometric invariant theory

ADVANTAGES

OF THE ALGEBRAIC APPROACH

1. simple algebraic equations instead of non-linear PDE system
2. third equation redundant!
3. valid for all (pseudo-)Riemannian constant curvature manifolds
4. Riemann tensors are intensively studied.
5. new insight into integrability from
   - representation theory
   - algebraic geometry
   - geometric invariant theory

Outline

Hodge decomposition
of algebraic curvature tensors

$$S^3 \subset V \qquad\qquad \dim V = 4$$

- algebraic curvature tensors as a symmetric 6×6 matrix

$$R_{a_1 b_1 a_2 b_2} \qquad \longleftrightarrow \qquad R^{a_1 b_1}{}_{a_2 b_2} \in \mathrm{End}(\Lambda^2 V)$$

- Hodge decomposition

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

- block decomposition

$$R = \left(\begin{array}{c|c} W_+ & T \\ \hline T^t & W_- \end{array}\right) + \frac{s}{12} I$$

$$W_+^T = W_+ \qquad W_-^T = W_- \qquad \operatorname{tr} W_+ = \operatorname{tr} W_- = 0$$

Hodge decomposition
of algebraic curvature tensors

$$S^3 \subset V \qquad \dim V = 4$$

- algebraic curvature tensors as a symmetric 6×6 matrix

$$R_{a_1 b_1 a_2 b_2} \longleftrightarrow R^{a_1 b_1}{}_{a_2 b_2} \in \mathrm{End}(\Lambda^2 V)$$

- Hodge decomposition

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

- block decomposition

$$R = \left(\begin{array}{c|c} W_+ & T \\ \hline T^t & W_- \end{array}\right) + \frac{s}{12} I$$

$$W_+^T = W_+ \qquad W_-^T = W_- \qquad \mathrm{tr}\, W_+ = \mathrm{tr}\, W_- = 0$$

Hodge decomposition

of algebraic curvature tensors

$$S^3 \subset V \qquad \dim V = 4$$

- algebraic curvature tensors as a symmetric 6×6 matrix

$$R_{a_1 b_1 a_2 b_2} \longleftrightarrow R^{a_1 b_1}{}_{a_2 b_2} \in \mathrm{End}(\Lambda^2 V)$$

- Hodge decomposition

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

- block decomposition

$$R = \left(\begin{array}{c|c} W_+ & T \\ \hline T^t & W_- \end{array} \right) + \frac{s}{12} I$$

$$W_+^T = W_+ \qquad W_-^T = W_- \qquad \mathrm{tr}\, W_+ = \mathrm{tr}\, W_- = 0$$

Hodge decomposition
of algebraic curvature tensors

$$S^3 \subset V \qquad\qquad \dim V = 4$$

- algebraic curvature tensors as a symmetric 6×6 matrix

$$R_{a_1 b_1 a_2 b_2} \qquad \longleftrightarrow \qquad R^{a_1 b_1}{}_{a_2 b_2} \in \mathrm{End}(\Lambda^2 V)$$

- Hodge decomposition

$$\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$$

- block decomposition

$$R = \left(\begin{array}{c|c} W_+ & T \\ \hline T^t & W_- \end{array}\right) + \frac{s}{12} I$$

$$W_+^T = W_+ \qquad W_-^T = W_- \qquad \mathrm{tr}\, W_+ = \mathrm{tr}\, W_- = 0$$

Integrable Killing tensors on S^3

as an algebraic variety

Theorem (–)

1. *K integrable* $\Rightarrow$ $\exists$ *orthonormal basis of V such that*

$$R = \begin{pmatrix} w_1-t_1 & & & & & \\ & w_2-t_2 & & & 0 & \\ & & w_3-t_3 & & & \\ & & & w_1+t_1 & & \\ & 0 & & & w_2+t_2 & \\ & & & & & w_3+t_3 \end{pmatrix} \qquad \Delta_1 := w_2 - w_3$$

2. *K integrable* $\Leftrightarrow$ $\det M = \operatorname{tr} M = 0$

$$M = \begin{pmatrix} \Delta_1 & -t_3 & +t_2 \\ +t_3 & \Delta_2 & -t_1 \\ -t_2 & +t_1 & \Delta_3 \end{pmatrix}$$

linear determinantal variety

$\mathcal{V} \subset \mathbb{P}^4$

INTEGRABLE KILLING TENSORS ON S^3

AS AN ALGEBRAIC VARIETY

THEOREM (–)

1. *K integrable* $\Rightarrow$ $\exists$ *orthonormal basis of V such that*

$$R = \begin{pmatrix} w_1 - t_1 & & & & & \\ & w_2 - t_2 & & & 0 & \\ & & w_3 - t_3 & & & \\ & & & w_1 + t_1 & & \\ & 0 & & & w_2 + t_2 & \\ & & & & & w_3 + t_3 \end{pmatrix} \qquad \Delta_1 := w_2 - w_3$$

2. *K integrable* $\Leftrightarrow$ $\det M = \operatorname{tr} M = 0$

$$M = \begin{pmatrix} \Delta_1 & -t_3 & +t_2 \\ +t_3 & \Delta_2 & -t_1 \\ -t_2 & +t_1 & \Delta_3 \end{pmatrix}$$

linear determinantal variety

$\mathcal{V} \subset \mathbb{P}^4$

Integrable Killing tensors on S^3

as an algebraic variety

Theorem (–)

1. *K integrable* $\Rightarrow$ $\exists$ *orthonormal basis of V such that*

$$R = \begin{pmatrix} w_1 - t_1 & & & & & 0 \\ & w_2 - t_2 & & & & \\ & & w_3 - t_3 & & & \\ & & & w_1 + t_1 & & \\ & & & & w_2 + t_2 & \\ 0 & & & & & w_3 + t_3 \end{pmatrix} \qquad \Delta_1 := w_2 - w_3$$

2. *K integrable* $\Leftrightarrow$ $\det M = \operatorname{tr} M = 0$

$$M = \begin{pmatrix} \Delta_1 & -t_3 & +t_2 \\ +t_3 & \Delta_2 & -t_1 \\ -t_2 & +t_1 & \Delta_3 \end{pmatrix}$$

linear determinantal variety

$\mathcal{V} \subset \mathbb{P}^4$

Stäckel systems

and separable coordinate systems

Theorem (Stäckel)

There is a bijective correspondence

$$\textit{Stäckel systems} \quad \longleftrightarrow \quad \textit{separable coordinate systems}$$

Definition

A Stäckel system is a vector space spanned by n linearly independent integrable Killing tensors which *mutually commute*.

Fact (–)

$$[K, \tilde{K}] = 0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1} \end{array}^{\star} g_{ij} R^i{}_{b_1 a_2 b_2} \tilde{R}^j{}_{d_1 c_2 d_2} = 0$$

STÄCKEL SYSTEMS

AND SEPARABLE COORDINATE SYSTEMS

THEOREM (STÄCKEL)

There is a bijective correspondence

$$\textit{Stäckel systems} \quad \longleftrightarrow \quad \textit{separable coordinate systems}$$

DEFINITION

A Stäckel system is a vector space spanned by n linearly independent integrable Killing tensors which *mutually commute*.

FACT (–)

$$\left[K, \tilde{K}\right] = 0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 & \multicolumn{3}{c}{} \\ \cline{1-1} c_2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}^{\,\star} g_{ij} R^{i}{}_{b_1 a_2 b_2} \tilde{R}^{j}{}_{d_1 c_2 d_2} = 0$$

STÄCKEL SYSTEMS

AS A SUBVARIETY OF A FANO VARIETY

FACT (–)

$$\left[K, \tilde{K}\right] = 0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 & \multicolumn{3}{c}{} \\ \cline{1-1} c_2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}^{\star} g_{ij} R^i{}_{b_1 a_2 b_2} \tilde{R}^j{}_{d_1 c_2 d_2} = 0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - ...
- simply check $(*)$ instead of solving $(*)$

Stäckel systems

as a subvariety of a Fano variety

Fact (–)

$$\left[K,\tilde{K}\right]=0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1}\end{array}^{\star} g_{ij}\,R^{i}{}_{b_1 a_2 b_2}\,\tilde{R}^{j}{}_{d_1 c_2 d_2}=0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V})\subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - …
- simply check $(*)$ instead of solving $(*)$

Stäckel systems

as a subvariety of a Fano variety

Fact (–)

$$\left[K, \tilde{K}\right] = 0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1}\end{array}^{\star} g_{ij} R^{i}{}_{b_1 a_2 b_2} \tilde{R}^{j}{}_{d_1 c_2 d_2} = 0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - ...
- simply check $(*)$ instead of solving $(*)$

STÄCKEL SYSTEMS
AS A SUBVARIETY OF A FANO VARIETY

FACT (–)

$$\left[K,\tilde{K}\right]=0 \qquad \Leftrightarrow \qquad \left(\begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 & \multicolumn{3}{c}{} \\ \cline{1-1} c_2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}\right)^{\star} g_{ij}\,R^i{}_{b_1 a_2 b_2}\,\tilde{R}^j{}_{d_1 c_2 d_2}=0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - ...
- simply check $(*)$ instead of solving $(*)$

Stäckel systems

as a subvariety of a Fano variety

Fact (–)

$$\left[K, \tilde{K}\right] = 0 \qquad \Leftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1}\end{array}^{\,\star} g_{ij} R^i{}_{b_1 a_2 b_2} \tilde{R}^j{}_{d_1 c_2 d_2} = 0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - ...
- simply check $(*)$ instead of solving $(*)$

Stäckel systems

as a subvariety of a Fano variety

Fact (–)

$$[K,\tilde{K}] = 0 \quad \Leftrightarrow \quad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1} \end{array}^{\star} g_{ij} R^{i}{}_{b_1 a_2 b_2} \tilde{R}^{j}{}_{d_1 c_2 d_2} = 0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - …
- simply check $(*)$ instead of solving $(*)$

Stäckel systems

as a subvariety of a Fano variety

Fact (–)

$$\left[K, \tilde{K}\right] = 0 \quad \Leftrightarrow \quad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & d_1 & d_2 \\ \hline a_2 \\ \cline{1-1} c_2 \\ \cline{1-1}\end{array}^{\star} g_{ij} R^{i}{}_{b_1 a_2 b_2} \tilde{R}^{j}{}_{d_1 c_2 d_2} = 0 \qquad (*)$$

- Stäckel systems $=$ projective lines on $\mathcal{V}$
- variety of projective lines on $\mathcal{V}$ $=$ Fano variety $F_1(\mathcal{V})$
- linear determinantal variety $\mathcal{V}$ $\subset$ full determinantal variety $\mathcal{M}$
- $F_1(\mathcal{V}) \subset F_1(\mathcal{M})$
- $F_1(\mathcal{M})$ well understood for 3×3 matrices
  - same kernel
  - same image
  - ...
- simply check $(*)$ instead of solving $(*)$

Stäckel systems
algebraically

Fact (–)

Stäckel systems correspond to projective lines of matrices

$$M = \begin{pmatrix} \Delta_1 & -t_3 & t_2 \\ t_3 & \Delta_2 & -t_1 \\ -t_2 & t_1 & \Delta_3 \end{pmatrix} \qquad \textit{with} \quad \operatorname{tr} M = \det M = 0$$

annihilating a fixed vector $v = (v_1, v_2, v_3)$.

- generically: projective line through

$$\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v_2^2 - v_3^2 & -v_1 v_2 & v_3 v_1 \\ v_1 v_2 & v_3^2 - v_1^2 & -v_2 v_3 \\ -v_3 v_1 & v_2 v_3 & v_1^2 - v_2^2 \end{pmatrix}$$

- $\mathcal{V}$ = join of projective plane and projected Veronese variety in $\mathbb{P}^4$

STÄCKEL SYSTEMS

ALGEBRAICALLY

FACT (–)

Stäckel systems correspond to projective lines of matrices

$$M = \begin{pmatrix} \Delta_1 & -t_3 & t_2 \\ t_3 & \Delta_2 & -t_1 \\ -t_2 & t_1 & \Delta_3 \end{pmatrix} \qquad \textit{with} \qquad \operatorname{tr} M = \det M = 0$$

annihilating a fixed vector $v = (v_1, v_2, v_3)$.

- generically: projective line through

$$\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v_2^2 - v_3^2 & -v_1 v_2 & v_3 v_1 \\ v_1 v_2 & v_3^2 - v_1^2 & -v_2 v_3 \\ -v_3 v_1 & v_2 v_3 & v_1^2 - v_2^2 \end{pmatrix}$$

- $\mathcal{V}$ = join of projective plane and projected Veronese variety in $\mathbb{P}^4$

Stäckel systems

algebraically

Fact (–)

Stäckel systems correspond to projective lines of matrices

$$M = \begin{pmatrix} \Delta_1 & -t_3 & t_2 \\ t_3 & \Delta_2 & -t_1 \\ -t_2 & t_1 & \Delta_3 \end{pmatrix} \qquad \textit{with} \quad \operatorname{tr} M = \det M = 0$$

annihilating a fixed vector $v = (v_1, v_2, v_3)$.

- generically: projective line through

$$\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v_2^2 - v_3^2 & -v_1 v_2 & v_3 v_1 \\ v_1 v_2 & v_3^2 - v_1^2 & -v_2 v_3 \\ -v_3 v_1 & v_2 v_3 & v_1^2 - v_2^2 \end{pmatrix}$$

- $\mathcal{V}$ = join of projective plane and projected Veronese variety in $\mathbb{P}^4$

Equivariant description

This completely solves our problem:

Theorem (–)

- *A generic Stäckel system consists of (multiples of)*
  - *Benenti-Killing tensors of the form*

$$\mathrm{Adj}(T+\lambda g) \owedge \mathrm{Adj}(T+\lambda g) \qquad \lambda \in \mathbb{R}$$

  - *and a unique trace free Ricci-Killing tensor*

$$T \owedge g$$

- *A non-generic Stäckel system is of the form*

$$\lambda_1 K_1 \odot *K_1 + \lambda_2 K_2 \odot *K_2 + \lambda_3 K_3 \odot *K_3$$

with $(\lambda_1, \lambda_2, \lambda_3) \perp \vec{n} \perp (1,1,1)$ *for some* $\vec{n} \in \mathbb{R}^3$

Equivariant description

This completely solves our problem:

Theorem (–)

- *A generic Stäckel system consists of (multiples of)*
  - *Benenti-Killing tensors of the form*

$$\mathrm{Adj}(T+\lambda g) \owedge \mathrm{Adj}(T+\lambda g) \qquad \lambda \in \mathbb{R}$$

  - *and a unique trace free Ricci-Killing tensor*

$$T \owedge g$$

- *A non-generic Stäckel system is of the form*

$$\lambda_1 K_1 \odot *K_1 + \lambda_2 K_2 \odot *K_2 + \lambda_3 K_3 \odot *K_3$$

with $(\lambda_1, \lambda_2, \lambda_3) \perp \vec{n} \perp (1,1,1)$ *for some* $\vec{n} \in \mathbb{R}^3$

Separable coordinates

- Jacobi elliptic coordinates
- Lamé rotational coordinates
- Lamé subgroup reduction
- spherical coordinates
- cylindrical coordinates

compare Eisenhart (1934) or Kalnins & Miller (1986)

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

GENERALISATIONS

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, . . .
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay   Geometry $\longleftrightarrow$ Algebra

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

GENERALISATIONS

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, . . .
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

Generalisations

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, . . .
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

Generalisations

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

Generalisations

- non-positive curvature
  - $\mathbb{R}^3$
  - $\mathbb{H}^3$: spinors (joint work with Robert Milson)
- higher dimensions
- non-constant curvature
  - $\mathbb{C}P^n$, Lie groups, symmetric spaces, ...
  - BGG? (joint work with Andreas Čap & Matthias Hammerl)
- other equations
- Moving Frames

interplay Geometry $\longleftrightarrow$ Algebra

Thanks for your attention!

R. G. McLenaghan, R. Milson & R. G. Smirnov:
"Killing tensors as irrepresentations of the general linear group"
CRAS Paris, 2004 (339) 621–624

K. Schöbel
"Algebraic Integrability Conditions for Killing Tensors
on Constant Sectional Curvature Manifolds"
http://lanl.arxiv.org/abs/1004.2872

K. Schöbel
"Separable Coordinates on the 3-Sphere"
in preparation