Holonomy reductions of Cartan connections and invariant differential equations

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- This talk is based on joint work with Rod Gover (Auckland) and Matthias Hammerl (Vienna).
- Cartan geometries give a conceptual description of manifolds endowed with certain geometric structures as "curved analogs" of a homogeneous space. They can also be interpreted as particularly nice reductions of higher order frame bundles.
- Holonomy reductions of a Cartan geometry can be defined parallel to the case of principal connections, but they exhibit a richer structure, in particular giving rise to a decomposition of the manifold in question into "curved orbits".
- Examples of such reductions can be obtained from parallel sections of so–called tractor bundles. In the special case of parabolic geometries, such parallel sections are related via the machinery of BGG sequences to solutions of certain geometric overdetermined systems.

Contents

Let G be a Lie group and $P \subset G$ a closed subgroup. Then the natural projection $p : G \to G/P$ is an P–principal bundle, and one has the left Maurer–Cartan form $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. The left actions of elements of G can be characterized as automorphisms of this principal bundle which are compatible with ω^{MC}. This motivates:

Definition

Let M be a smooth manifold with $\dim(M) = \dim(G/P)$. A *Cartan geometry* of type (G, P) on M is a principal P–bundle $p : \mathcal{G} \to M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.

1. $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism $\forall u \in \mathcal{G}$.
2. For the principal right action r^g by $g \in P$ we have $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$.
3. For the fundamental vector field ζ_X generated by $X \in \mathfrak{p}$ we have $\omega(\zeta_X) = X$.

Relation to frame bundles

Via the adjoint action, P acts on $\mathfrak{g}$ and $\mathfrak{p} \subset \mathfrak{g}$ is a P–invariant subspace. Thus there is an induced action $\underline{\mathrm{Ad}} : P \to GL(\mathfrak{g}/\mathfrak{p})$. Let K be the kernel of this homomorphism and put $\underline{P} := P/K$.

- $\mathcal{G}/K \to M$ is a principal $\underline{P}$–bundle.
- projecting the values of ω to $\mathfrak{g}/\mathfrak{p}$, the resulting form descends to a strictly horizontal, $\underline{P}$–equivariant one–form $\theta \in \Omega^1(\mathcal{G}/K, \mathfrak{g}/\mathfrak{p})$ and $\mathfrak{g}/\mathfrak{p} = \mathbb{R}^{\dim(M)}$.
- Thus we obtain an induced first order structure on M with structure group $\underline{P}$.

More generally, one associates to $g \in P$ the k–jet at $o = eP$ of the left action of g. If this map is injective for some k, this makes P into a subgroup of the kth jet group. One shows that, via ω, $\mathcal{G}$ defines a reduction of the kth order frame bundle of M.

Examples

(1) For $n \geq 2$ put $G = Euc(n) = O(n) \ltimes \mathbb{R}^n$ and $P = O(n) \subset G$. Then G/P is Euclidean space, $\mathfrak{g}/\mathfrak{p} = \mathbb{R}^n$ and $\underline{\text{Ad}}$ is the standard representation of $O(n)$ on $\mathbb{R}^n$. Hence a Cartan geometry $(\mathcal{G} \to M, \omega)$ induces a first order $O(n)$–structure, which is equivalent to a Riemannian metric on M. The $\mathfrak{o}(n)$ component of ω defines a metric connection on TM, from which ω can be recovered.

Via the orthonormal frame bundle and the Levi–Civita connection, one can conversely associate a Cartan geometry of type (G, P) to any Riemannian metric on M. This induces an equivalence of categories between Riemannian manifolds and torsion free Cartan geometries.

(2) Put $G = PGL(n+1,\mathbb{R})$ and let $P \subset G$ be the stabilizer of a line in $\mathbb{R}^{n+1}$, so $P \cong GL(n,\mathbb{R}) \ltimes \mathbb{R}^{n*}$. Then $G/P = \mathbb{R}P^n$, $\mathfrak{g}/\mathfrak{p} = \mathbb{R}^n$, $P/K = GL(n,\mathbb{R})$ and $\underline{\mathrm{Ad}}$ is the standard representation. Thus the first order structure underlying a Cartan geometry of type (G,P) contains no information. Such a Cartan geometry turns out to be equivalent to a projective equivalence class of linear connections on TM.

The subgroup $P \subset G$ in this example is *parabolic* in the sense of representation theory. For parabolic subgroups in semisimple groups there is a general theory, initiated by N. Tanaka, of equivalence of Cartan geometries (satisfying a condition on the curvature) to underlying structures. ("Parabolic Geometries") This leads to a description of conformal structures, hypersurface type CR structures, almost quaternionic structures, path geometries, quaternionic contact structures, and several types of generic distributions as Cartan geometries.

Remarks

- The theorems establishing existence of canonical Cartan connections and equivalence of categories between Cartan geometries and underlying structures are often difficult and technically demanding.
- The curvature $K = d\omega + \frac{1}{2}[\omega, \omega]$ of the Cartan connection is a basic and complete invariant of a Cartan geometry.
- I want to view this just as the beginning of the story, and explain some of the things one can do once one has a canonical Cartan connection available, in particular in the parabolic case. Hence the description as a Cartan geometry will always be considered as being given in the sequel.

Holonomy of Cartan connections

Since $\omega : T_u\mathcal{G} \to \mathfrak{g}$ is injective, there are no curves in $\mathcal{G}$ which are parallel for ω. To still get a notion of holonomy, one can connect to the classical case of principal connections:

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) then:

- $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$ is a G–principal bundle
- $\exists!$ G–principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $\tilde{\omega}|_{T\mathcal{G}} = \omega$.
- One then defines $\mathrm{Hol}(\omega) := \mathrm{Hol}(\tilde{\omega}) \subset G$.

Recall that $\mathrm{Hol}(\tilde{\omega}) \subset G$ is defined up to conjugacy only. While this is harmless for principal connections it becomes a very important issue here, since a conjugation does not fix the subgroup P in general.

holonomy reductions

In view of the conjugacy issue, it is better not to use a subgroup of G to define a holonomy reduction but rather a homogeneous space $\mathcal{O}$ of G. (For example, one would use the space of inner products on $\mathbb{R}^n$ instead of $O(n) \subset GL(n, \mathbb{R})$ to describe the holonomy reductions of a linear connection given by a parallel metric.)

- Form the associated bundle $\mathcal{G} \times_P \mathcal{O}$.
- Since this can be viewed as $\tilde{\mathcal{G}} \times_G \mathcal{O}$, it inherits a canonical (non–linear) connection from $\tilde{\omega}$.
- A *holonomy reduction of type* $\mathcal{O}$ of a Cartan geometry $(\mathcal{G} \to M, \omega)$ is a section of $\mathcal{G} \times_P \mathcal{O}$ which is parallel for this induced connection.

By general principles, sections of $\mathcal{G} \times_P \mathcal{O} \cong \tilde{\mathcal{G}} \times_G \mathcal{O}$ can be either identified with P–equivariant functions $\mathcal{G} \to \mathcal{O}$ or with G–equivariant functions $\tilde{\mathcal{G}} \to \mathcal{O}$. In the latter picture, a function corresponds to a parallel section if and only if it is constant along any curve which is horizontal for $\tilde{\omega}$. This can be used to clarify holonomy reductions of the homogeneous model:

- The bundle $G \times_P G$ is canonically trivial via $(g_1, g_2) \mapsto (g_1 P, g_1 g_2)$ and $\tilde{\omega}$ is the flat connection induced by this trivialization.
- Hence any element $\alpha \in \mathcal{O}$ determines a unique holonomy reduction of $(G \to G/P, \omega^{MC})$ of type $\mathcal{O}$ corresponding to the equivariant function $G \to \mathcal{O}$ defined by $s_\alpha : G \to \mathcal{O}$, $s_\alpha(g) := g^{-1} \cdot \alpha$, and any reduction is of this type.

P–types and curved orbit decomposition

Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) and consider a holonomy reduction of type $\mathcal{O}$ corresponding to the equivariant function $s : \mathcal{G} \to \mathcal{O}$.
Then for any $x \in M$, the image $s(\mathcal{G}_x) \subset \mathcal{O}$ of the fiber $\mathcal{G}_x = p^{-1}(x)$ is a P–orbit in $\mathcal{O}$, called the *P–type of x*.

- Let $\mathcal{O} = \cup \mathcal{O}_i$ be the decomposition of $\mathcal{O}$ into P–orbits.
- Then there is a corresponding decomposition $M = \cup M_i$ according to P–types. (Some of the M_i may be empty.)
- For the reduction s_α of the homogeneous model constructed before, consider the stabilizer $G_\alpha \subset G$ of α. Then $g_1 P, g_2 P \in G/P$ have the same P–type if and only if they lie in the same G_α–orbit. In particular, on may identify $\mathcal{O}/P$ with the space of G_α–orbits in G/P.

The properties of the decomposition $M = \cup M_i$ are studied using a local diffeomorphism (the "comparison map") between the curved geometry and an appropriate holonomy reduction of $G \to G/P$. This is constructed using normal coordinates induced by Cartan connections. One shows:

- The comparison map intertwines the decompositions $M = \cup M_i$ and $G/P = \cup (G/P)_i$. In particular, each M_i is an initial submanifold of M.
- For $\alpha \in \mathcal{O}_i$ let $G_\alpha \subset G$ be its stabilizer and put $P_\alpha := G_\alpha \cap P$. Then M_i inherits a canonical Cartan geometry of type (G_α, P_α) generalizing $G_\alpha \to G_\alpha/P_\alpha = G_\alpha \cdot eP \subset G/P$.
- The curvature of this induced geometry can be explicitly described in terms of the curvature of ω.

A simple way to define a holonomy reduction of a principal bundle is by a parallel section of an associated vector bundle. In the case of a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), we have to use a vector bundle of the form $\mathcal{G} \times_P \mathbb{V} = \tilde{\mathcal{G}} \times_G \mathbb{V}$, for a representation $\mathbb{V}$ of G ("tractor bundles").

- Such sections can be either described by P–equivariant functions $\mathcal{G} \to \mathbb{V}$ of by G–equivariant functions $\tilde{\mathcal{G}} \to \mathbb{V}$.
- In the latter picture, parallel sections correspond to functions which are constant along $\tilde{\omega}$–horizontal curves.
- For a parallel section s, the image of $\tilde{\mathcal{G}}$ in $\mathbb{V}$ hence is a G–orbit $\mathcal{O} \subset \mathbb{V}$ ("G–type of s"), which defines the type of holonomy reduction determined by s.
- The images of the fibers of $\mathcal{G}$ in $\mathcal{O}$ are P–orbits, which define the P–types of points.

In the case that G is semisimple and $P \subset G$ is parabolic, one may assume that $\mathbb{V}$ is an irreducible representation of G. Then $\mathbb{V}$ inherits a canonical P–invariant filtration $\mathbb{V} \supset \mathbb{V}^1 \supset \cdots \supset \mathbb{V}^N$, such that $\mathbb{V}^i/\mathbb{V}^{i+1}$ is a completely reducible representation of P for each i and $\mathbb{V}/\mathbb{V}^1 =: \mathbb{H}_0$ is even irreducible.

- The bundle $\mathcal{H}_0 := \mathcal{G} \times_P \mathbb{H}_0$ is naturally a quotient of $\mathcal{G} \times_P \mathbb{V}$, so any section s of the tractor bundle induces $\Pi(s) \in \Gamma(\mathcal{H}_0)$.
- If s is parallel, then $\Pi(s)$ lies in the kernel of a natural differential operator defined on $\Gamma(\mathcal{H}_0)$ which gives rise to a geometric overdetermined system.
- In this way, one obtains twistor spinors and almost Einstein scales in conformal geometry, special conformal Killing forms and Killing tensors, and special infinitesimal automorphisms for all parabolic geometries.

Consider a parallel section s of $\mathcal{G} \times_P \mathbb{V}$ of G–type $\mathcal{O}$. Since $\mathbb{V}^1 \subset \mathbb{V}$ is P–invariant, also $\mathcal{O} \cap \mathbb{V}^1$ and its complement in $\mathcal{O}$ are P–invariant and hence a union of P–types. By construction, for $x \in M$, we have $\Pi(s)(x) = 0$ if and only if the P–type of s at x lies in $\mathcal{O} \cap \mathbb{V}^1$, so the zero set of $\Pi(s)$ is a union of P–types.

- Via comparison, s is related to a (local) parallel section of $G \times_P \mathbb{V} \to G/P$. The latter sections can be easily described explicitly.
- Hence the zero set of $\Pi(s)$ cannot look worse than the one of this model section.
- The P–types provide a stratification of the zero set of $\Pi(s)$ which again can look at most as complicated as the one for the model section.

Ricci flat connections

Put $G := SL(n+1,\mathbb{R})$, $P \subset G$ the stabilizer of a ray in $\mathbb{R}^{n+1}$ (so one gets oriented projective structures) and $\mathbb{V} := \mathbb{R}^{(n+1)*}$. Then $\mathbb{V} \setminus 0$ is one G–orbit, which splits into three P–orbits. Two of these are open, one is closed and coincides with $\mathbb{V}^1 \setminus 0$. For a parallel section s, the underlying section $\sigma := \Pi(s)$ is a density, which satisfies a natural second order equation, and one gets:

- The curved orbit decomposition has the form $M = M_+ \cup M_0 \cup M_-$ with $M_\pm \subset M$ open and M_0 an embedded hypersurface which is the zero set of σ.
- On $M_\pm$, σ determines a Ricci flat connection in the projective class, which has some completeness property.
- The hypersurface $M_0 \subset M$ is totally geodesic and thus inherits a projective structure.

Fefferman spaces

Put $G := SO(2p+2, 2q+2)$, P the stabilizer of an isotropic line (so one gets conformal Riemannian structures of signature $(2p+1, 2q+1)$). In $\mathbb{V} := \mathfrak{so}(2p+2, 2q+2)$ there is the G–orbit $\mathcal{O}$ of orthogonal complex structures on $\mathbb{R}^{2p+2,2q+2}$.

- For a parallel section s of $\mathcal{G} \times_P \mathbb{V}$ of this G–type, the underlying section $\Pi(s)$ is a special conformal Killing field k on M.
- Since $\mathcal{O}$ is also a single P–orbit, k is nowhere vanishing and thus defines a one–dimensional foliation of M.
- One then proves that a local leaf space of this foliation inherits a CR structures and M is locally conformally isometric to the Fefferman space of this CR structure.

Almost Einstein scales

Consider conformal structures of signature (p, q), so $G = SO(p+1, q+1)$ and $P \subset G$ the stabilizer of an isotropic line and $\mathbb{V} = \mathbb{R}^{p+1,q+1}$. Then $\mathcal{H}_0$ is a density bundle. G–types in $\mathbb{V}$ are the level sets of $\langle v, v \rangle$ for the G–invariant inner product on $\mathbb{V}$, and the essential bit is the sign of $\langle v, v \rangle$.

- Any parallel section $s \in \Gamma(\mathcal{G} \times_P \mathbb{V})$ has constant norm $h(s, s)$ for the tractor metric h and $\sigma = \Pi(s)$ satisfies the conformally invariant equation $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab} \sigma = 0$.
- For $h(s, s) \neq 0$, the curved orbit decomposition is $M = M_0 \cup M_1$ with $M_0 = \mathcal{Z}(\sigma)$ an embedded hypersurface.
- On M_1, σ determines an Einstein metric in the conformal class, while M_0 inherits a canonical conformal structure, and locally around M_0 one obtains a Ponicaré–Einstein metric.

Klein-Einstein structures

This is an example for projective structures, which produces results which are similar in flavor to (but different from) Poincaré–Einstein metrics.

Take $G = SL(n+1,\mathbb{R})$, $P \subset G$ the stabilizer of a ray in $\mathbb{R}^{n+1}$ and $\mathbb{V} = S^2\mathbb{R}^{(n+1)*}$. Then $\mathcal{H}_0$ is a density bundle, and G–types are determined by rank and signature.

- If s is a parallel section, whose G–type is non–degenerate, then $\sigma = \Pi(s)$ satisfies the projectively invariant equation

$$\nabla_{(a}\nabla_b\nabla_{c)}\sigma + 4\mathrm{P}_{(ab}\nabla_{c)}\sigma + 2(\nabla_{(a}\mathrm{P}_{bc)})\sigma = 0$$

- P–types are determined by rank and signature of the restriction of the metric determined by s to the distinguished line subbundle in $\mathcal{G} \times_P \mathbb{R}^{n+1}$.

- The curved orbit decomposition has the form $M = M_+ \cup M_0 \cup M_-$ with $M_\pm$ open and M_0 an embedded hypersurface.
- On $M_\pm$, σ determines a connection ∇ in the projective class, for which P_{ab} is symmetric, non–degenerate and satisfies $\nabla_a \mathrm{P}_{bc} = 0$. Hence P_{ab} defines a pseudo–Riemannian metric (whose signature is determined by the signature of s) with Levi–Civita connection ∇, and which must be Einstein.
- If M is compact, then these Einstein metrics on $M_\pm$ are geodesically complete.
- M_0 canonically inherits a conformal structure (whose signature is determined by the signature of s).