All 3D and 4D Lorentzian manifolds of maximal order

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Preliminaries

- Smooth, connected, n-dimensional Lorentzian manifold (M, g)
- Lorentzian inner product $(\mathbb{R}^n, \eta)$ with $\eta_{ab} = \eta_{ba},\ a, b = 1, \ldots, n$
  η-orthogonal coframe: isomorphism $(T_x M, g_x) \to (\mathbb{R}^n, \eta)$
- Group of η-orthogonal transformations: $O(\eta) = \mathrm{Aut}(\eta)$
  Lie algebra $\mathfrak{o}(\eta)$ with basis $A_\alpha,\ \alpha = 1, \ldots, n(n-1)/2$.
- Principal $O(\eta)$-bundle of η-orthogonal coframes $O(\eta, M) \to M$
  Moving η-orthogonal coframe: section $\boldsymbol{\omega} : U \to O(\eta, U),\ U \subset M$.
  Equivalently, moving coframe ω^a is η-orthogonal if $g = \eta_{ab}\omega^a \otimes \omega^b$
- Canonical lifted coframe: $(\hat{\omega}^a, \hat{\Gamma}^\alpha)$ subject to structure equations:

$$\mathrm{d}\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{\Gamma}} \wedge \boldsymbol{\omega} = 0, \quad \mathrm{d}\hat{\boldsymbol{\Gamma}} + \hat{\boldsymbol{\Gamma}} \wedge \hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{\Omega}}$$

- Let $\boldsymbol{R}$ denote the curvature tensor as a section of $\otimes^4 T^* M$.
  Let $\hat{R}_{abcd} : O(\eta, M) \to \mathbb{R}$ be the corresponding lift:

$$\hat{\Omega}^\alpha A_{\alpha ab} = \frac{1}{2} \hat{R}_{abcd}\, \hat{\omega}^c \wedge \hat{\omega}^d$$

  Let $\hat{R}_{abcd;e_1 \ldots e_p} : O(\eta, M) \to \mathbb{R}$ be the lift of $\nabla^p \boldsymbol{R}$:

$$\mathrm{d}\hat{R}_{abcd;e_1 \ldots e_{p-1}} = \hat{R}_{abcd;e_1 \ldots e_p} \hat{\omega}^{e_p} + (A_\alpha \cdot \hat{R})_{abcd} \hat{\Gamma}^\alpha$$

Equivalence problem for metric tensors

- Set $\mathcal{R}^p = \otimes^4(\mathbb{R}^n)^* \oplus \cdots \oplus \otimes^{4+p}(\mathbb{R}^n)^*$
  Let $\hat{R}^{(p)} : O(\eta, M) \to \mathcal{R}^p$ be the composite lift of $\boldsymbol{R}, \nabla \boldsymbol{R}, \ldots, \nabla^p \boldsymbol{R}$:

$$\hat{R}^{(p)} = (\hat{R}_{abcd}, \hat{R}_{abcd;e}, \ldots, \hat{R}_{abcd;e_1\ldots e_p})$$

- Say that (M, g) is *regular* if the rank of $\hat{R}^{(p)}$ is constant for all p.
  Let $\rho_p := \operatorname{rank} \hat{R}^{(p)}$. Note: $\rho_p \leq \rho_{p+1} \leq \dim O(\eta, M) = n(n+1)/2$
  Let $q = q_M$ be the smallest integer such that $\rho_{q-1} = \rho_q$.
  Call $q - 1$ *the order* of (M, g).
- **Theorem [Cartan,Olver]:** Suppose that (M, g) is regular. Then $\hat{R}^{(q)} : O(\eta, M) \to \mathcal{R}^{(q)}$ parameterizes a classifying manifold.
  Manifolds (M, g) and $(\tilde{M}, \tilde{g})$ are locally isometric iff $q_M = q_{\tilde{M}}$ and $\hat{R}^{(q)}$ and $\hat{\tilde{R}}^{(q)}$ parameterize overlapping classifying manifolds.
- **Proposition:** $q_M \leq n(n+1)/2 = \dim O(\eta, M)$.
  Question: for a fixed signature and dimension, what is the maximum order?

Main result - 4 dimensions

Theorem (Milson,Pelavas): for 4-dimensional Lorentzian manifolds, $q \leq 7$. The bound is sharp. All order 6 metrics that don't pseudo-stabilize are locally equivalent to $g = \omega^1\omega^2 - \omega^3\omega^4$ where

$$\omega^1 = \left(db/2 + i(a\,db + da) - e^b\left(a + i\left(a^2 + C_2/2 - 5/4\right)\right)ds\right)/C_1,$$

$$\omega^2 = \left(db/2 - i(a\,db + da) - e^b\left(a - i\left(a^2 + C_2/2 - 5/4\right)\right)ds\right)/C_1,$$

$$\omega^3 = e^b ds,$$

$$\omega^4 = e^{-3b}dt - (C_2/C_1^2)(da + a\,db) + \\ + \left(F(s)e^{-3b} - 6\,ae^{-2b}t + (C_2/C_1^2)\left(a^2 + C_2/4 - 5/8\right)e^b\right)ds.$$

Note 1: the above metric is an exact solution for coupled gravity and null-radiation on an anti-deSitter background.

Note 2: there may or may not exist order 6 metrics that pseudo-stabilize (ongoing research).

Main result - 3 dimensions

Theorem (Milson, Wylleman): For 3-dimensional Lorentz manifolds, $q \leq 5$. The above bound is sharp. All order 4 metrics, up to local equivalence, belong to one of the following 3 classes:

$$(ds_i)^2 = 2\,(x\,du + dw)^2 + du\left(f_i\,du - T^{-1}dx\right), \quad i = 1, 2, \quad T \neq 0,$$

$$(ds_3)^2 = 2dw^2 + 2du((\phi(u) + 2v + Cx^2)du - dx), \quad C \neq 0,$$

where C, K, T are real constants, where $\phi(u)$ is a real-valued function, and where

$$f_1 = KT^{-1}(e^{4w} - 1) + \frac{1}{2}(x + \phi(u))\,((K-2)x + (K+2)\phi(u)) - \phi'(u),$$

$$f_2 = -x^2 + |T|^{-5/2}x + |T|^{-1}e^{4Tw} + \phi(u)$$

Talk overview

- Joint work with Nicos Pelavas (4D) and Lode Wylleman (3D)
- Equivalence problem: Cartan (1926, 1946), Karlhede (1980)
- Curvature classification: Petrov (1954,1961), Penrose (1960),
- Curvature homogeneity: Singer (1960), Bueken, Boeckx, Djoric, Gilkey, Kovalski, VanHecke (1996,2000,2008)
- The Karlhede algorithm
- Curvature homogeneity
- Petrov type
- Pseudo-stabilization and other technicalities

Exact Solutions of Einstein's Field Equations

Second Edition

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1. Set the order of differentiation q to 0.
2. Calculate the derivatives of the Riemann tensor up to the qth.
3. Find the canonical form of the Riemann tensor and its derivatives.
4. Fix the frame as far as possible by this canonical form, and note the residual frame freedom (the group of allowed transformations is the linear isotropy group $\hat{I}_q$). The dimension of $\hat{I}_q$ is the dimension of the remaining vertical part of the frame bundle.
5. Find the number t_q of independent functions of space-time position in the components of the Riemann tensor and its derivatives in canonical form. This tells us the remaining horizontal freedom.
6. If the isotropy group and number of independent functions are the same as at the previous step, let $p+1=q$ and stop; if they differ (or if $q=0$) increment q by 1 and go to step 2.

The space-time is then characterized by the canonical form used, the successive isotropy groups and independent function counts and the values of the non-zero Cartan invariants. Since there are t_p essential space-time coordinates, clearly the remaining $4-t_p$ are ignorable, so the isotropy group of the space-time will have dimension $s=\dim \hat{I}_p$ and the isometry group has dimension $r=s+4-t_p$ (see e.g. Karlhede (1980b)). To com-

Curvature normalization and regularity

- **Definitions.** Let $S \subset \mathcal{R}^{(p)}$ be a submanifold, set $G = O(\eta)$ and let $H = G_S$, meaning that $H = \{X \in G : X(S) \subset S\}$.
  Say that C is an H-kernel if H is a closed subgroup of G and if the mapping $S \times_H G \to G(S)$ is a diffeomorphism.
  Say that S is a *normalizing cross-section* if it is an H-kernel and if in addition H fixes S point-wise.
  Note: If we added the assumption that $G(S)$ is an open subset of $\mathcal{R}^{(p)}$, then S would be a slice [Palais].
- **Proposition.** Let $S \subset \mathcal{R}^{(p)}$ be an H-kernel where $H = G_S, G = O(\eta)$ as above. Suppose that the image of $\hat{R}^{(p)}$ is contained in $G(S)$. Then

$$O(\eta, M, S) := (\hat{R}^{(p)})^{-1}(S)$$

  is a principal H-bundle over M, and the embedding $O(\eta, M, S) \hookrightarrow O(\eta, M)$ is a reduction of structure from G to H.
  Moreover, $\hat{R}^{(p)}|O(\eta, M, S)$ is constant in the fibres and descends to a well-defined mapping $R_S^{(p)} : M \to S$ (called Cartan invariants).
  Equivalently, $R_S^{(p)} = \hat{R}^{(p)}|\boldsymbol{\omega} = (R_{abcd}, R_{abcd;e}, \ldots, R_{abcd;e_1\ldots e_p})$ is independent of the choice of $\boldsymbol{\omega} : U \to O(\eta, U, S),\ U \subset M$.
- Example: diagonalize the Ricci tensor to find a preferred coframe (maximal reduction of the structure group).

The Karlhede algorithm

- **Definition.** Call (M, g) *p-curvature regular* if there exists a normalizing cross-section $S \subset \mathcal{R}^p$ such that img $\hat{R}^{(p)} \subset G(S)$, where $G = O(\eta)$.
- **Proposition:** Suppose that (M, g) is p-regular and p-curvature regular. Let $R_S^{(p)} : M \to S$ be the normalized curvature mapping into a normalizing cross-section $S \subset \mathcal{R}^p$. Then, $\tau_p := \text{rank}\, R_S^{(p)}$ is constant and independent of the choice of S. Moreover, setting $\nu_p = \dim G_S$,

$$\rho_p = \tau_p + \dim G - \dim G_S = \tau_p + n(n-1)/2 - \nu_p$$

  Moreover, if $q - 1$ is the order of (M, g), then $R_S^{(q)} : M \to S$ is a rank τ_q parameterization of a classifying manifold for (M, g).
  The integer $n - \tau_q + \nu_q$ is the dim. of the isometry group, with ν_q the dim. of the isotropy subgroup and $n - \tau_q$ the dim. of the orbits.
- **Algorithm:** $S^{(0)} \to S^{(1)} \to \cdots \to S^{(q)}$ a tower of normalizing cross-sections with structure groups $G^{(p)} = G_{S^{(p)}}^{(p-1)}$ and $G^{(-1)} = O(\eta)$.
  Integer seqences: $\nu_p = \dim G^{(p)}$ (non-increasing) and $\tau_p = \text{rank}\, R_{S^{(p)}}^{p}$ (non-decreasing) with q the smallest integer such that $\nu_{q-1} = \nu_q$ and $\tau_{q-1} = \tau_q$.

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Infinitesimally Homogeneous Spaces*

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Introduction

The results of this paper were motivated by this question: How alike must two points of a Riemannian manifold be in order to conclude that the manifold is homogeneous, i.e., that there is a transitive group of isometries of the manifold? One must of course assume that the manifold is complete,

In this paper we eliminate the local Lie group hypothesis and in fact prove homogeneity under an infinitesimal rather than a local hypothesis. Specifically there exists an integer k such that if for every two points the curvature and its first k covariant derivatives at m_1 and m_2 are the same, then the manifold is homogeneous. If the dimension of the manifold is d, it turns out that $k < \frac{1}{2}d(d-1)$. At any rate, since a local isometry preserves the curvature and *all* its derivatives, we obtain as a corollary that a weakly locally homogeneous space is homogeneous (see the Corollary of the Main Theorem).

Curvature homogeneity

- **Definition.** Say that (M, g) is *k-curvature homogeneous* (CH_k) if for every $x, y \in M$ there exists an isomorphism $\phi : (T_xM, g_x) \to (T_yM, g_y)$ such that
$$\phi^* \nabla^j \boldsymbol{R}_y = \nabla^j \boldsymbol{R}_x, \quad j = 0, 1, \ldots, k$$
We say that (M, g) is *properly* CH_k if it is CH_k, but not CH_{k+1}.
- **Equivalent definition 1**: there exists a cover by moving coframes $\boldsymbol{\omega} : U \to O(\eta, U)$ with constant $\hat{R}^{(k)}|\boldsymbol{\omega} = (R_{abcd}, R_{abcd;e}, \ldots, R_{abcd;e_1\ldots e_k})$
- **Equivalent definition 2**: CH_k iff $\tau_k = 0$.
Properly CH_k iff k is the smallest integer for which $\tau_{k+1} \neq 0$.
- **Theorem [Singer]** A manifold is locally homogeneous iff it is curvature homogeneous, i.e., CH_k for all k.
- **Definition**: for a fixed dimension and signature, the *Singer number* is the maximum value of k such that there exists a proper CH_k. In other words, the test for homogeneity requires
$\hat{R}^{(k)} = (\hat{R}_{abcd}, \hat{R}_{abcd;e}, \ldots, \hat{R}_{abcd;e_1\ldots e_{k+1}})$ where k is the Singer number.
- Observation: the manifolds of maximal order turn out to be proper CH_k manifolds that realize the Singer number. For such manifolds we have
$$\nu_0 > \nu_1 > \cdots > \nu_k = \nu_{k+1} = 0, \quad 0 = \tau_0 = \tau_1 = \cdots = \tau_k < \tau_{k+1}.$$
- **Concusion:** The Singer number is the key to manifolds of maximum order. Classify proper CH_k where k is as large as possible.

The 3D classification

Theorem: The Singer number for 3-dimensional Lorentzian geometry is $k = 1$. The unique (up to a 3D-Lorentz transformation) proper CH_1 geometries satisfy the following structure equations:

$$\begin{aligned}
\mathrm{d}\omega^0 &= -T\omega^0 \wedge \omega^1 \\
\mathrm{d}\omega^1 &= -4T\omega^0 \wedge \omega^2 \\
\mathrm{d}\omega^2 &= \omega^0 \wedge \omega^1 + 2\epsilon\omega^0 \wedge \omega^2 - T\omega^1 \wedge \omega^2 \\
\mathrm{d}\epsilon \wedge \omega^0 &= \epsilon T\omega^0 \wedge \omega^1 + C\omega^0 \wedge \omega^2
\end{aligned}$$

Note 1: the metric is $g = -2\omega^0\omega^2 + \frac{1}{2}(\omega^1)^2$. The curvature components $(R_{abcd}, R_{abcd;e})$ are functions of the constants T, C.
Note 2: rank $R^{(2)} = 1$ with 2nd order curvature scalars $(R_{abcd;e_1e_2})$ functions of C, T, ϵ where the latter is a differential invariant with $\mathrm{d}\epsilon \neq 0$.
Note 3: By direct calculation: rank $R^{(3)} = 2$, rank $R^{(4)} = 3$. Therefore (wonderful coincidence!) the proper CH_1 metrics have maximal order.
Note 3: the 3 classes of exact solution correspond to the cases (i) $T, C \neq 0$; (ii) $T \neq 0, C = 0$; (iii) $T = 0, C \neq 0$.

The 4D classification

Theorem: The Singer number for 4-dimensional Lorentzian geometry is $k = 2$. The unique (up to Lorentz transformation) proper CH_2 geometries satisfy the following structure equations:

$$\begin{aligned}
\mathrm{d}\omega^1 &= C_1\,\omega^1\wedge\omega^2 - \mathrm{i}(C_2-3)\,(\omega^1+\omega^2)\wedge\omega^3,\\
\mathrm{d}\omega^2 &= -C_1\,\omega^1\wedge\omega^2 + \mathrm{i}(C_2-2)\,(\omega^1+\omega^2)\wedge\omega^3,\\
\mathrm{d}\omega^3 &= C_1\,(\omega^1+\omega^2)\wedge\omega^3,\\
\mathrm{d}\omega^4 &= -2\mathrm{i}C_2\,\omega^1\wedge\omega^2 - \mathrm{i}\nu(\omega^1-\omega^2)\wedge\omega^3 - 3C_1\,(\omega^1+\omega^2)\wedge\omega^4,\\
\mathrm{d}\nu\wedge\omega^3 &= \left((3\mathrm{i}/2)\,C_2\,(\omega^1-\omega^2) - 3C_1\nu\,(\omega^1+\omega^2) - 3C_1\omega^4\right)\wedge\omega^3.
\end{aligned}$$

Note 1: the metric is $g = \omega^1\omega^2 - \omega^3\omega^4$ where ω^1, ω^2 are complex conjugate, while ω^3, ω^4 are real. The curvature components $(R_{abcd}, R_{abcd;e}, R_{abcd;e_1e_2})$ are functions of the real constants C_1, C_2.
Note 2: rank $R^{(3)} = 1$ with third order curvature scalars $(R_{abcd;e_1e_2e_3})$ functions of C_1, C_2, ν, where the latter is a scalar invariant with $\mathrm{d}\nu \neq 0$.
Note 3: By direct calculation: rank $R^{(4)} = 2$, rank $R^{(5)} = 3$, rank $R^{(6)} = 4$.
Therefore (wonderful coincidence!) the proper CH_2 metrics have maximal order.

The Karlhede bound and the 4D Petrov type

- The Penrose-Petrov classification: orbit structure of the Weyl tensor in 4-dimensional Lorentzian geometry
- $G_0 = \mathrm{Aut}(R^{(0)})$ where $R^{(0)} = \hat{R}^{(0)}|O(\eta, M, S)$ with S a normalizing c.s.
- Rank formula: $\rho_0 = \tau_0 + \dim O(\eta) - \dim G_0$
  rank lower bound: $\rho_0 \geq n(n-1)/2 - \nu_0$
  Karlhede bound: $n(n+1)/2 \geq \rho_{q-1} \geq \rho_0 + q - 1$

$$q - 1 \leq n + \nu_0$$

- Double cover $\mathrm{SL}_2\mathbb{C} \to SO(1,3)$.
  As irreps, $\mathrm{Weyl}_4 \cong \mathcal{P}_4\mathbb{C}$ (complex polynomials of degree ≤ 4).

Petrov Type	Root Structure	ν_0	Karlhede bound	Singer number
I	(1111)	0	$q \leq 5$	$k \leq 0$
II	(211)	0	$q \leq 5$	$k \leq 0$
III	(31)	0	$q \leq 5$	$k \leq 0$
D	(22)	2	$q \leq 7$	$k \leq 2$
N	(4)	2	$q \leq 7$	$k \leq 2$
O	...	3	$q \leq 8$	$k \leq 2$

- The search for maximal order manifolds and the Singer number narrow to Petrov type D, N, O. The type O bound comes from automorphism groups of the Ricci tensor; this possibility is easily eliminated.

The 3D Petrov type

- Double cover $\mathrm{SL}_2\mathbb{R} \to SO(1,2)$.
  As irreps, $\mathrm{Riem}_3 \cong \mathrm{Sym}^2\,\mathbb{R}^3 \cong 1 \oplus \mathcal{P}_4\mathbb{R}$ (real polynomials of degree ≤ 4).

Petrov Type	Root Structure	Segre Type	ν_0	$q \leq$	$k \leq$
I	(1111)	$\{11,1\}$	0	4	0
IZ	$(Z\bar{Z}11)$	$\{1z\bar{z}\}$	0	4	0
IZZ	$(Z_1Z_2\bar{Z}_1\bar{Z}_2)$	$\{11,1\}$	0	4	0
II	(211)	$\{2,1\}$	0	4	0
IIZ	$(2Z\bar{Z})$	$\{2,1\}$	0	4	0
III	(31)	$\{(11),1\}$	0	4	0
D	(22)	$\{1(1,1)\}$	1	5	1
DZ	$(Z^2\bar{Z}^2)$	$\{3\}$	1	5	1
N	(4)	$\{(21)\}$	1	5	1
O	$\dots$				

- The search for maximal order manifolds and the Singer number narrow to Petrov types D, DZ N

Some Technicalities

- **Karlhede Algorithm:** $S^{(0)} \to S^{(1)} \to \cdots \to S^{(q)}$ a tower of normalizing cross-sections with structure groups $G^{(p)} = G^{(p-1)}_{S^{(p)}}$ and $G^{(-1)} = O(\eta)$. Integer seqences: $\nu_p = \dim G^{(p)}$ (non-increasing) and $\tau_p = \text{rank}\, R^p_{S^{(p)}}$ (non-decreasing) with q the smallest integer such that $\nu_{q-1} = \nu_q$ and $\tau_{q-1} = \tau_q$.
- **Pseudo-stabilization**: $\nu_p = \nu_{p+1}, \tau_p < \tau_{p+1}$
- A re-formulation:
  - normalize R_{abcd} to reduce the structure group from $G_{-1} = O(\eta)$ to $G_0 = \text{Aut}(R^{(0)})$.
  - this reduction gives a constrained, overdetermined equivalence problem. The essential torsion consists of certain connection scalars, which also figure in the components of $R_{abcd;e}$.
  - if the essential torsion is G_0 invariant, the structure group cannot be reduced. However, if functionally independent invariants arise, the equivalence method does not terminate.

General structure equations

$$(\eta_{ab}) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \qquad \left(\hat{\Gamma}^a{}_b\right) = \begin{pmatrix} \hat{\Gamma}_{02} & \hat{\Gamma}_{12} & 0 \\ -2\hat{\Gamma}_{01} & 0 & 2\hat{\Gamma}_{12} \\ 0 & -\hat{\Gamma}_{01} & -\hat{\Gamma}_{02} \end{pmatrix}$$

$$\begin{aligned} \mathrm{d}\hat{\omega}^0 &= -\hat{\Gamma}_{02} \wedge \hat{\omega}^0 - \hat{\Gamma}_{12} \wedge \hat{\omega}^1 \\ \mathrm{d}\hat{\omega}^1 &= 2\hat{\Gamma}_{01} \wedge \hat{\omega}^0 + 2\hat{\Gamma}_{12} \wedge \hat{\omega}^2 \\ \mathrm{d}\hat{\omega}^2 &= \hat{\Gamma}_{01} \wedge \hat{\omega}^1 + \hat{\Gamma}_{02} \wedge \hat{\omega}^2 \end{aligned} \qquad \begin{aligned} \mathrm{d}\hat{\Gamma}_{01} &= -\hat{\Gamma}_{01} \wedge \hat{\Gamma}_{02} + \hat{\Omega}_{01} \\ \mathrm{d}\hat{\Gamma}_{02} &= -2\,\hat{\Gamma}_{01} \wedge \hat{\Gamma}_{12} + \hat{\Omega}_{02} \\ \mathrm{d}\hat{\Gamma}_{12} &= -\hat{\Gamma}_{02} \wedge \hat{\Gamma}_{12} + \hat{\Omega}_{12} \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_{01} &= \frac{1}{2}\Psi_0\,\hat{\omega}^0 \wedge \hat{\omega}^1 + \Psi_1\,\hat{\omega}^0 \wedge \hat{\omega}^2 + (\Psi_2/2 + R/12)\,\hat{\omega}^1 \wedge \hat{\omega}^2 \\ \hat{\Omega}_{02} &= \Psi_1\,\hat{\omega}^0 \wedge \hat{\omega}^1 + (2\Psi_2 - R/6)\,\hat{\omega}^0 \wedge \hat{\omega}^2 + \Psi_3\hat{\omega}^1 \wedge \hat{\omega}^2 \\ \hat{\Omega}_{12} &= (\Psi_2/2 + R/12)\,\hat{\omega}^0 \wedge \hat{\omega}^1 + \Psi_3\,\hat{\omega}^0 \wedge \hat{\omega}^2 + \frac{1}{2}\Psi_4\,\hat{\omega}^1 \wedge \hat{\omega}^2 \end{aligned}$$

Note 1: the Bianchi relations not shown
Note 2: R is the Ricci scalar. The curvature components transform like

$$\Psi(z) = \Psi_0 + \Psi_1 4z + \Psi_2 6z^2 + \Psi_3 4z^3 + \Psi_4 z^4$$

under the action of $\mathrm{SL}_2\mathbb{R}$

The type D, CH_0 reduction

The normalization: $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$; R, Ψ_2 are invariant constants.
Rank: $\rho_0 = 2,\ \nu_0 = 1,\ \tau_0 = 0$
Reduction (after applying integrability conditions):

$$\Gamma_{01} = \kappa\omega^0 + \tau\omega^2, \qquad \Gamma_{12} = \tau\omega^0 + \nu\omega^2$$

Reduced structure equations:

$$\begin{aligned}
\mathrm{d}\omega^0 &= -\Gamma_{02} \wedge \omega^0 - (\tau\omega^0 + \nu\omega^2) \wedge \omega^2 \\
\mathrm{d}\omega^1 &= -4\tau\omega^0 \wedge \omega^2 \\
\mathrm{d}\omega^2 &= \Gamma_{02} \wedge \omega^2 + (\kappa\omega^0 + \tau\omega^2) \wedge \omega^1 \\
\mathrm{d}\Gamma_{02} &= (2\Psi_2 - R/6 - 2\kappa\nu + 2\tau^2)\omega^0 \wedge \omega^2 \\
\mathrm{d}\kappa \wedge \omega^0 + \mathrm{d}\tau \wedge \omega^2 &= 2\kappa\Gamma_{02} \wedge \omega^0 + (\Psi_2/2 + R/12 - \kappa\nu + \tau^2)\omega^1 \wedge \omega^2 \\
\mathrm{d}\tau \wedge \omega^0 + \mathrm{d}\nu \wedge \omega^2 &= -2\nu\Gamma_{02} \wedge \omega^2 + (\Psi_2/2 + R/12 - \kappa\nu + \tau^2)\omega^0 \wedge \omega^1
\end{aligned}$$

Note 1: Bianchi relations are satisfied identically
Note 2: the remaining freedom is the 1-dimensional group of boosts (diagonal SL_2 transformation)

$$\omega^{0\prime} = a^{-1}\omega^0, \quad \omega^{1\prime} = \omega^1, \quad \omega^{2\prime} = a\omega^2, \quad a \neq 0$$

$$\kappa' = a^2\kappa, \quad \nu' = a^{-2}\nu, \quad \tau' = \tau$$

Type D, CH_1 reduction

- Generic subcase: $\kappa, \nu \neq 0$. Normalize $\kappa \to 1$.
  The CH_1 constraint: τ, ν constant
  After imposing integrability conditions: $\rho_1 = \rho_2 = 3,\ \nu_1 = 0,\ \tau_1 = 0$

  $$d\omega^0 = -(\tau\omega^0 + \nu\omega^2) \wedge \omega^1 \quad d\omega^1 = -4\tau\omega^0 \wedge \omega^2 \quad d\omega^2 = (\omega^0 + \tau\omega^2) \wedge \omega^1$$

  where τ, ν are constants. Invariant metric on a Lie group.
- Singular subcase: $\kappa \neq 0, \nu = 0$. Normalize $\kappa \to 1$. The CH_1 constraint: τ constant Note: $\nu \leftrightarrow \kappa$ by a Lorentz transformation
  Rank: $\rho_0 = 2, \rho_1 = 3, \rho_2 = 4, \nu_1 = 0, \tau_1 = 0, \tau_2 = 1$
  After applying integrability conditions get the CH_1 structure equations of the 3D Theorem. Generically, these metrics have order 4 (maximal)
- Stabilization subcase: $\kappa = \nu = 0$
  After imposing integrability conditions, pseudo-stabilization isn't possible
  Get a 0th order homogeneous space w/ 1d isotropy
  $\rho_0 = \rho_1 = 2,\ \nu_0 = \nu_1 = 1,\ \tau_0 = \tau_1 = 0$

  $$d\omega^0 = -\Gamma_{02} \wedge \omega^0 - \tau\omega^0 \wedge \omega^1 \quad d\omega^1 = -4\tau\omega^0 \wedge \omega^2$$
  $$d\omega^2 = \Gamma_{02} \wedge \omega^2 + \tau\omega^2 \wedge \omega^1, \quad d\Gamma_{02} = (3\Psi_2 + 4\tau^2)\omega^0 \wedge \omega^2$$
  $$R = -6\Psi_2 - 12\tau^2, \quad \Psi_2, \tau \text{ constant}$$

Concluding remarks

- The Karlhede bound is sharp for 3D and 4D Lorentzian geometry ($q = 5$ and $q = 7$) respectively.
- The corresponding maximal order manifolds can be classified explicitly (a bit more work remains to be done in the 4D case)
- Multiple communities investigating equivalence of metric tensors: Cartan equivalence method, GR/Karlhede algorithm, curvature homogeneity/Singer number.
- The key requirement is an understading of the orbit structure of the Rieman curvature tensor. At $n \geq 4$, the orbit structure of the Weyl tensor representation is the key object.
- Curvature normalization reduces the structure group and permits further analysis.
- There is an important connection between the Singer number and maximal order. Metrics that realize the Singer number are natural candidates for the maximal order condition.