Equivalence of geometric structures in control theory via moving frames

Workshop on Moving Frames in Geometry
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The method of moving frames grew out of Felix Klein's Erlangen program, which proposed that the study of geometry is really about the study of invariants of geometric objects under group actions. Examples include:

- Geometry of submanifolds of $\mathbb{R}^n$ under the action of the Euclidean group consisting of rotations, reflections, and translations;
- Geometry of submanifolds of $\mathbb{R}^n$ under other group actions (e.g., affine, conformal, projective);
- Geometry of submanifolds of other homogeneous spaces G/H (e.g., $S^n, \mathbb{H}^n, \mathbb{P}^n$) under the action of G.

Associated to each object of interest is a collection of invariants which are preserved by the group action. Familiar examples of invariants under the Euclidean group include:

- Curvature and torsion for a unit-speed curve in $\mathbb{R}^3$;
- Gauss and mean curvatures for a surface in $\mathbb{R}^3$.

Invariants provide a necessary condition for equivalence: two objects cannot be equivalent under the group action unless their invariants agree.

The ultimate goal is to find a *complete* set of invariants for the objects of interest under the group action. If a complete set of invariants can be found, then they provide a necessary *and* sufficient condition for equivalence: two objects are equivalent under the group action if and only if their invariants agree.

The method of moving frames was introduced by Élie Cartan in order to address this problem. In this method, a principal bundle B (called a *frame bundle*) is associated to each n-dimensional object M under consideration; each fiber of the bundle

$$\pi : B \to M$$

is isomorphic to a fixed Lie group G representing the appropriate geometry.

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Each local section

$$\sigma : M \to B$$

of the a frame bundle represents a choice of a *local framing* on M. Generally, for each $\mathbf{x} \in M$, $\sigma(\mathbf{x})$ is a framing of the tangent space $T_{\mathbf{x}}M$ which is somehow *adapted* to the geometry of M.

The fiber group G is a subgroup of $GL(n)$, chosen so that the action of G preserves the set of adapted frames.

Then the equivalence problem becomes:

Given principal bundles B_1, B_2 over manifolds M_1, M_2, respectively, when does there exist a diffeomorphism $\Phi : M_1 \to M_2$ such that

$$\Phi_*(B_1) = B_2?$$

Under certain conditions (but not always!), Cartan's theory can give a complete answer to this question.

Example: Riemannian surfaces. To any Riemannian surface M, we can associate the *orthonormal frame bundle* $\mathcal{F}(M)$, whose fiber at each point $\mathbf{x} \in M$ consists of all orthonormal frames $\mathbf{f} = (\mathbf{e}_1, \mathbf{e}_2)$ for the tangent space $T_{\mathbf{x}}M$. $\mathcal{F}(M)$ is a principal bundle with fiber group $O(2)$.

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Given two Riemannian surfaces M_1, M_2 and an isometry

$$\Phi : M_1 \to M_2,$$

Φ_* acts on orthonormal frames in the obvious way: if $\mathbf{f} = (\mathbf{e}_1, \mathbf{e}_2)$ is an orthonormal frame based at $\mathbf{x} \in M_1$, then $\Phi_*(\mathbf{f})$ is the frame

$$\Phi_*(\mathbf{f}) = (\Phi_*(\mathbf{e}_1), \Phi_*(\mathbf{e}_2))$$

based at the point $\Phi(\mathbf{x}) \in M_2$.

Moreover, a diffeomorphism $\Phi : M_1 \to M_2$ is an isometry if and only if

$$\Phi_*(\mathcal{F}(M_1)) = \mathcal{F}(M_2).$$

So the equivalence problem,

"When are two Riemannian surfaces M_1, M_2 isometric?"

can be reformulated as,

"When does there exist a diffeomorphism $\Phi : M_1 \to M_2$ such that

$$\Phi_*(\mathcal{F}(M_1)) = \mathcal{F}(M_2)?"$$

A frame bundle $\pi : B \to M$ carries a canonical vector-valued 1-form ω, defined by the property that for any $(\mathbf{x}, \mathbf{f}) \in B$ and any $\mathbf{v} \in T_{(\mathbf{x},\mathbf{f})}B$,

$$\omega(\mathbf{v}) = \pi_*(\mathbf{v}) \in T_{\mathbf{x}}M.$$

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If $\mathbf{f} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a framing for the tangent space $T_{\mathbf{x}}M$, then ω can be written as

$$\omega = \mathbf{e}_i \, \omega^i,$$

where $\omega^1, \ldots, \omega^n$ are scalar-valued 1-forms on B. These are called the *dual forms* (also known as the *solder forms*) on B; they have the property that for any local section $\sigma : M \to B$, the pullbacks

$$\bar{\omega}^i = \sigma^* \omega^i$$

form a local coframing on M which is dual to the local framing on M defined by σ.

The dual forms are also *semi-basic* for the projection $\pi : B \to M$. This means that for any $\mathbf{v} \in T_{(\mathbf{x},\mathbf{f})}B$ which is tangent to the fiber $B_{\mathbf{x}}$ over the point $\mathbf{x} \in X$, we have

$$\omega^i(\mathbf{v}) = 0$$

for $i = 1, \ldots, n$.

Theorem . *Let* B_1*,* B_2 *be principal bundles over manifolds* M_1*,* M_2*, respectively, with canonical* 1*-forms* ω*,* Ω*. Then there exists a diffeomorphism* $\Phi : M_1 \to M_2$ *such that*

$$\Phi_*(B_1) = B_2$$

if and only if there exists a diffeomorphism

$$\Phi^{(1)} : B_1 \to B_2$$

such that

$$\left(\Phi^{(1)}\right)^* \Omega = \omega.$$

Invariants for the equivalence problem on a principal bundle $B \to M$ appear in the *structure equations* for the dual forms, which express the exterior derivatives of the dual forms as

$$d\omega^i = \alpha^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k,$$

where the matrix

$$\alpha = [\alpha^i_j]$$

is a $\mathfrak{g}$-valued 1-form and the T^i_{jk} are functions on B, called *torsion functions*.

The method of equivalence proceeds by using the group action to normalize the torsion functions as much as possible; this results in a reduction of the structure group to a smaller group, which then introduces additional torsion functions, and so on.

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Ideally, this process leads to a reduced frame bundle whose fiber group is the trivial group $G = (e)$. In this case, the frame bundle is called an (e)-*structure* on M, and the torsion functions in the structure equations, together with their derivatives up to some fixed order, provide a *complete* set of invariants for the equivalence problem.

The method of equivalence can be applied to a wide variety of problems in differential equations as well as geometry; in particular, it can be used to compute invariants of systems of differential equations under various types of coordinate transformations.

These invariants can be used to address questions such as: "When is a given PDE system equivalent to a linear system under an appropriate change of variables?"

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These invariants can be used to address questions such as: "When is a given PDE system equivalent to a linear system under an appropriate change of variables?"

In this talk we will see some examples arising in control theory, which may be viewed as the study of underdetermined ODE systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}).$$

Equivalence for oriented Riemannian surfaces

Given an oriented Riemannian surface M, let $\mathcal{F}_+(M)$ denote the *oriented* orthonormal frame bundle of M. $\mathcal{F}_+(M)$ is a principal bundle over M with fiber $SO(2)$.

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Remark: $\mathcal{F}_+(M)$ may also be thought of as the unit circle bundle of M, via the identification

$$(\mathbf{e}_1, \mathbf{e}_2) \leftrightarrow \mathbf{e}_1.$$

The dual forms (ω^1, ω^2) on $\mathcal{F}_+(M)$ are characterized by the conditions that:

- ω^1, ω^2 are semi-basic for the projection $\pi : \mathcal{F}_+(M) \to M$.
- For any section $\sigma : M \to \mathcal{F}_+(M)$, the pullbacks $\bar{\omega}^i = \sigma^*(\omega^i)$ to M satisfy
$$\bar{\omega}^i(\mathbf{e}_j(x)) = \delta^i_j;$$
i.e., ω^i measures the component of a tangent vector to M in the $\mathbf{e}_i$ direction.
- $g = (\omega^1)^2 + (\omega^2)^2$.

Furthermore, there exists a unique 1-form α on $\mathcal{F}_+(M)$ which is linearly independent from ω^1, ω^2, and satisfies

$$\begin{aligned} d\omega^1 &= -\alpha \wedge \omega^2 \\ d\omega^2 &= \alpha \wedge \omega^1. \end{aligned} \tag{1}$$

α is called the *connection form* on $\mathcal{F}_+(M)$; it is the Levi-Civita connection for the Riemannian metric on M.

It has the further property that

$$d\alpha = K\,\omega^1 \wedge \omega^2, \tag{2}$$

where K is the Gauss curvature of the Riemannian metric on M.

Together, equations (1) and (2) form the structure equations for the canonical coframing $(\omega^1, \omega^2, \alpha)$ on $\mathcal{F}_+(M)$.

Cartan's general theory implies that the Gauss curvature K of M, together with its covariant derivatives K_1, K_2, defined by the condition

$$dK = K_1\,\omega^1 + K_2\,\omega^2,$$

form a complete set of local invariants for the Riemannian metric on M.

Subtle point: The equivalence problem begins by considering the bundle $\mathcal{F}_+(M)$ of adapted frames over the surface M. The algorithm leads us to the canonical coframing $(\omega^1, \omega^2, \alpha)$, which is defined upstairs on $\mathcal{F}_+(M)$, rather than on M.

Technically, the frame bundle $\mathcal{F}_+(M) \to M$ has been replaced by a new frame bundle

$$\mathcal{F}_+^{(1)}(M) \to \mathcal{F}_+(M),$$

called the *prolongation* of $\mathcal{F}_+(M) \to M$, with the old frame bundle as the new base manifold. In this case, the structure group of the new frame bundle is the identity $G = (e)$, and so

$$\mathcal{F}_+^{(1)}(M) \cong \mathcal{F}_+(M).$$

Static equivalence for kinematic control systems

A *kinematic control system* may be described in local coordinates by an underdetermined system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$ represents the *state* of the system and $\mathbf{u} \in \mathbb{R}^s$ represents the *controls*. More generally, $\mathbf{x}$ and $\mathbf{u}$ may take values in an n-dimensional manifold X and an s-dimensional manifold U, respectively.

The system is *controllable* if, given any two states $\mathbf{x}_1, \mathbf{x}_2$, there exists a solution curve of (3) connecting $\mathbf{x}_1$ to $\mathbf{x}_2$. Such a solution curve is called an *admissible path* in X.

Remark: The adjective "kinematic" refers to the fact that the control variables are used to control the first derivatives of the state variables. A *dynamic* control system is one in which the control variables are used to control the second derivatives of the state variables; this is commonly the case in control of mechanical systems, where controls take the form of external forces.

The most commonly considered equivalence problem for control systems is that of *static* (or *feedback*) equivalence. Two control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{4}$$

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{v}) \tag{5}$$

on state spaces X, Y, respectively, are called *static equivalent* if there exists a diffeomorphism

$$\Phi : X \to Y$$

such that for every solution curve $\mathbf{x}(t)$ of (4), the curve

$$\mathbf{y}(t) = \Phi(\mathbf{x}(t))$$

is a solution curve of (5), and vice-versa.

Geometric interpretation: The system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

defines a submanifold Σ of TX consisting of all admissible values for $\dot{\mathbf{x}}$: for each $\mathbf{x} \in X$, the fiber $\Sigma_{\mathbf{x}}$ is defined by

$$\Sigma_{\mathbf{x}} = \{\mathbf{f}(\mathbf{x}, \mathbf{u}) \,|\, \mathbf{u} \in U\}.$$

Static equivalence for the two systems (4), (5), represented by submanifolds

$$\Sigma_1 \subset TX, \qquad \Sigma_2 \subset TY,$$

respectively, is equivalent to the condition that there exists a diffeomorphism

$$\Phi : X \to Y$$

such that

$$\Phi_*(\Sigma_1) = \Sigma_2.$$

Control-affine and control-linear systems

The system (3) is called *control-affine* if $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is affine linear in the control variables; in this case (3) takes the form

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{s} \mathbf{f}_i(\mathbf{x}) u^i. \tag{6}$$

The vector field $\mathbf{f}_0(\mathbf{x})$ is called the *drift* vector field; it determines the dynamics of the system in the absence of controls.

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Fact: The first prolongation of *any* control system is control-affine.

A system for which $\mathbf{f}_0(\mathbf{x}) \equiv \mathbf{0}$ is called a *driftless* system; we will also refer to such a system as *control-linear*. In this case we can write (6) as

$$\dot{\mathbf{x}} = A(\mathbf{x})\mathbf{u}, \tag{7}$$

where $A(\mathbf{x})$ is an $n \times s$ matrix.

If $A(\mathbf{x})$ has constant rank s, then the associated submanifold $\Sigma \subset TX$ is a rank s (linear) distribution $\mathcal{D}$ on X, and we can regard the variables $(\mathbf{x}, \mathbf{u})$ as local coordinates on the distribution $(X, \mathcal{D})$.

By a theorem of Chow, the system (7) is controllable if and only if the distribution $\mathcal{D}$ on X is *bracket-generating*, i.e., if the iterated brackets of vector fields contained in $\mathcal{D}$ span the entire tangent space at each point $\mathbf{x} \in X$.

So the static equivalence problem for controllable, control-linear kinematic systems may be reformulated as an equivalence problem for bracket-generating distributions $(X, \mathcal{D})$ under the group of diffeomorphisms of the base manifold X.

Geometry of linear distributions in low dimensions

Associated to a bracket-generating distribution $\mathcal{D}$ is a flag of distributions

$$\mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^r = TX,$$

called the *derived flag* of $\mathcal{D}$, defined by

$$\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \qquad i \geq 1.$$

If all the $\mathcal{D}^i$ have constant rank, we say that $\mathcal{D}$ has *constant type*. The *growth vector* of a constant type distribution $\mathcal{D}$ is the list of integers

$$(\mathrm{rank}(\mathcal{D}^1), \ldots, \mathrm{rank}(\mathcal{D}^r)).$$

The classical theorems of Frobenius, Pfaff, and Engel imply that for $n \leq 4$, distributions of constant type are classified up to local equivalence by their growth vectors.

For instance, the generic rank 2 distribution $\mathcal{D}$ on a 4-manifold X has growth vector $(2, 3, 4)$, and Engel's theorem implies that every point $\mathbf{x} \in X$ has a neighborhood on which there exist local coordinates (x^1, x^2, x^3, x^4) such that

$$\mathcal{D} = \left\{ \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3}, \ \frac{\partial}{\partial x^4} \right\}.$$

A 4-manifold X together with a rank 2 distribution $\mathcal{D}$ of this type is called an *Engel manifold*.

Beginning with $n = 5$ and $s = 2$, local invariants depending on arbitrary functions appear: Cartan's famous paper, *"Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre,"* describes local invariants of rank 2 distributions on 5-manifolds with growth vector $(2, 3, 5)$.

More recently, Bryant has described local invariants of rank 3 distributions on 6-manifolds with growth vector $(3, 6)$, and Doubrov and Zelenko have given a fairly comprehensive treatment of maximally nonholonomic distributions of ranks 2 and 3 on manifolds of arbitrary dimension.

Things are more interesting in low dimensions if we consider the problem of *optimal control*: given a distribution $(X, \mathcal{D})$ representing a system of the form (7), what is the most efficient horizontal path between two given points in X? In order to answer this question, we must have some measure of the energy required to move in the state space.

Typically, this is specified by a first-order Lagrangian functional L: for any horizontal curve $\gamma : [a, b] \to X$, we define the *action integral*

$$\mathcal{L}(\gamma) = \int_\gamma L(\mathbf{x}, \dot{\mathbf{x}})\, d\mathbf{x} = \int_\gamma \bar{L}(\mathbf{x}, \mathbf{u})\, d\mathbf{x},$$

where we set $\bar{L}(\mathbf{x}, \mathbf{u}) = L(\mathbf{x}, A(\mathbf{x})\mathbf{u})$.

Often the Lagrangian is given by the square root of a smoothly varying, positive-definite inner product on each subspace $\mathcal{D}_{\mathbf{x}}$; i.e.,

$$\bar{L}(\mathbf{x}, \mathbf{u}) = \sqrt{g_{ij}(\mathbf{x}) u^i u^j}.$$

In this case, the Lagrangian defines a *sub-Riemannian metric* g on $\mathcal{D}$ (i.e., a Riemannian metric on each subspace $\mathcal{D}_{\mathbf{x}} \subset T_{\mathbf{x}} X$). The triple $(X, \mathcal{D}, g)$ is called a *sub-Riemannian manifold.*

Horizontal paths which minimize the action integral are the *geodesics* of the sub-Riemannian metric.

Equivalence for sub-Riemannian contact 3-manifolds (Hughen)

A *contact* 3-*manifold* is a 3-manifold X equipped with a rank 2 distribution $\mathcal{D}$ of constant type, with growth vector $(2,3)$; such a distribution is called a *contact distribution* on X.

Pfaff's theorem implies that every point $\mathbf{x}$ of a contact 3-manifold X has a neighborhood on which there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{D} = \left\{ \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2},\ \frac{\partial}{\partial x^3} \right\}.$$

Let $(X, \mathcal{D}, g)$ be an oriented sub-Riemannian structure on a contact 3-manifold $(X, \mathcal{D})$. Consider the bundle $\mathcal{F}_0 \to X$ of *adapted frames* for the structure $(X, \mathcal{D}, g)$, whose fiber over each point $\mathbf{x} \in X$ consists of all frames $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for $T_{\mathbf{x}}X$ with the properties that:

- $\mathcal{D}_{\mathbf{x}} = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$;
- $(\mathbf{e}_1, \mathbf{e}_2)$ form an oriented, orthonormal frame for the metric g on $\mathcal{D}_{\mathbf{x}}$.

The method of equivalence leads to a reduction of $\mathcal{F}_0$ to:

- an $SO(2)$-bundle $\Sigma \to X$, which may be thought of as the "unit circle bundle" for g via the identification

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \leftrightarrow \mathbf{e}_1;$$

- a canonical coframing $(\omega^1, \omega^2, \omega^3, \alpha)$ on Σ that satisfies the structure equations:

$$\begin{aligned} d\omega^1 &= -\alpha \wedge \omega^2 + A_1\, \omega^2 \wedge \omega^3 + A_2\, \omega^3 \wedge \omega^1 \\ d\omega^2 &= \alpha \wedge \omega^1 + A_2\, \omega^2 \wedge \omega^3 - A_1\, \omega^3 \wedge \omega^1 \\ d\omega^3 &= \omega^1 \wedge \omega^2 \\ d\alpha &= S_1\, \omega^2 \wedge \omega^3 + S_2\, \omega^3 \wedge \omega^1 + K\, \omega^1 \wedge \omega^2. \end{aligned} \tag{8}$$

Differentiating these equations shows that S_1, S_2 may be expressed in terms of the covariant derivatives of A_1 and A_2; therefore, the functions A_1, A_2, K, together with their covariant derivatives, form a complete set of local invariants for the sub-Riemannian structure $(X, \mathcal{D}, g)$ under the group of diffeomorphisms of the base manifold X.

Now consider: What if the natural energy measure on horizontal curves is not given by a quadratic form? It is not difficult to imagine examples where the dependence of $\bar{L}$ on $\mathbf{u}$ gets quite complicated as $\mathbf{u}$ changes direction; some natural Lagrangians might not even satisfy the symmetry condition

$$\bar{L}(\mathbf{x}, -\mathbf{u}) = \bar{L}(\mathbf{x}, \mathbf{u}).$$

This leads us to generalize the notion of a sub-Riemannian metric on $(X, \mathcal{D})$ by replacing the Riemannian metric on each subspace $\mathcal{D}_{\mathbf{x}} \subset T_{\mathbf{x}}X$ with a Finsler metric.

Definition: A *Finsler metric* on a manifold M is a function

$$F : TM \to [0, \infty)$$

with the following properties:

- Regularity: F is C^∞ on the slit tangent bundle $TM \setminus 0$.
- Positive homogeneity: $F(\mathbf{x}, \lambda\mathbf{y}) = \lambda F(\mathbf{x}, \mathbf{y})$ for all $\lambda > 0$. (Here $\mathbf{x}$ is any system of local coordinates on M and $(\mathbf{x}, \mathbf{y})$ is the corresponding canonical coordinate system on TM.)
- Strong convexity: The $n \times n$ matrix

$$\left[\frac{\partial^2(\frac{1}{2}F^2)}{\partial y^i\,\partial y^j}\right]$$

is positive definite at every point of $TM \setminus 0$.

Replacing the Riemannian metric g on each subspace $\mathcal{D}_{\mathbf{x}} \subset T_{\mathbf{x}}X$ with a Finsler metric F leads to a *sub-Finsler metric* on $(X, \mathcal{D})$. The triple $(X, \mathcal{D}, F)$ is called a *sub-Finsler manifold*.

The problem of finding action-minimizing paths satisfying (7) is then equivalent to finding geodesics of the sub-Finsler manifold $(X, \mathcal{D}, F)$.

Replacing the Riemannian metric on $\mathcal{D}$ by a Finsler metric allows much more general energy functionals to be considered. The requirement that the Lagrangian be the square root of a quadratic form is replaced by the more natural requirements that:

- it be positive-homogeneous of degree one in $\mathbf{u}$ (which is necessary if the length of an oriented curve is to be independent of parametrization)
- it be strongly convex (which is necessary if there are to exist locally minimizing paths in every direction).

Equivalence for sub-Finsler contact 3-manifolds (C—, Moseley)

Let $(X, \mathcal{D}, F)$ be an oriented sub-Finsler structure on a contact 3-manifold $(X, \mathcal{D})$. Unlike in Riemannian geometry, there is no notion akin to an "orthonormal frame" in Finsler geometry.

Instead of defining a principal bundle over X, we must work on the *indicatrix bundle* Σ of $(X, \mathcal{D}, F)$, defined by the property that

$$\Sigma = \{\mathbf{u} \in \mathcal{D} \mid F(\mathbf{u}) = 1\}.$$

Σ is a 4-dimensional manifold, and the equivalence problem begins by defining adapted frames on Σ rather than on X.

The method of equivalence produces a canonical coframing $(\eta^1, \eta^2, \eta^3, \phi)$ on Σ that satisfies the structure equations:

$$
\begin{aligned}
d\eta^1 &= -\phi \wedge \eta^2 + A_1\, \eta^2 \wedge \eta^3 + (A_2 + \tfrac{1}{2} I K)\, \eta^3 \wedge \eta^1 + J_1\, \phi \wedge \eta^3 \\
d\eta^2 &= \phi \wedge \eta^1 + (A_2 - \tfrac{1}{2} I K)\, \eta^2 \wedge \eta^3 - A_1\, \eta^3 \wedge \eta^1 + J_2\, \phi \wedge \eta^3 \\
&\qquad - I\, \phi \wedge \eta^2 \\
d\eta^3 &= \eta^1 \wedge \eta^2 - I\, \phi \wedge \eta^3 \\
d\phi &= S_0\, \eta^3 \wedge \phi + S_1\, \eta^2 \wedge \eta^3 + S_2\, \eta^3 \wedge \eta^1 \\
&\qquad - J_1\, \phi \wedge \eta^1 - 2 J_2\, \phi \wedge \eta^2 + K\, \eta^1 \wedge \eta^2.
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d\eta^2 &= \phi \wedge \eta^1 + (A_2 - \tfrac{1}{2} I K)\,\eta^2 \wedge \eta^3 - A_1\,\eta^3 \wedge \eta^1 + J_2\,\phi \wedge \eta^3 \\
&\qquad - I\,\phi \wedge \eta^2 \\
d\eta^3 &= \eta^1 \wedge \eta^2 - I\,\phi \wedge \eta^3 \\
d\phi &= S_0\,\eta^3 \wedge \phi + S_1\,\eta^2 \wedge \eta^3 + S_2\,\eta^3 \wedge \eta^1 \\
&\qquad - J_1\,\phi \wedge \eta^1 - 2J_2\,\phi \wedge \eta^2 + K\,\eta^1 \wedge \eta^2.
\end{aligned}
$$

Theorem (C— -Moseley)**.** *The sub-Finsler structure represented by the indicatrix bundle* Σ *is sub-Riemannian if and only if* $I \equiv 0$.

Similar results have been obtained for Engel manifolds:

- Sub-Riemannian case: Moseley
- Sub-Finsler case: C—, Moseley, and Wilkens

Control-affine systems and affine distributions

For the control-affine system

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{s} \mathbf{f}_i(\mathbf{x})u^i,$$

the associated submanifold $\Sigma \subset TX$ is a rank s *affine distribution* $\mathcal{F}$ whose fiber $\mathcal{F}_{\mathbf{x}}$ over each point $\mathbf{x} \in X$ is defined by

$$\mathcal{F}_{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \operatorname{span}\{\mathbf{f}_1(\mathbf{x}), \ldots, \mathbf{f}_s(\mathbf{x})\}.$$

Static equivalence for two control-affine systems

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{s} \mathbf{f}_i(\mathbf{x})u^i, \tag{9}$$

$$\dot{\mathbf{y}} = \mathbf{g}_0(\mathbf{y}) + \sum_{i=1}^{s} \mathbf{g}_i(\mathbf{y})v^i \tag{10}$$

on state spaces X, Y, respectively, represented by affine distributions $\mathcal{F}_1 \subset TX$, $\mathcal{F}_2 \subset TY$, is equivalent to the condition that there exists a diffeomorphism $\Phi : X \to Y$ such that

$$\Phi_*(\mathcal{F}_1) = \mathcal{F}_2.$$

In this case, we say that the affine distributions $\mathcal{F}_1, \mathcal{F}_2$ are *affine equivalent*.

Equivalence for affine distributions under affine equivalence was studied by Elkin, who found that functional invariants appear in lower dimensions than for linear distributions: if $\mathcal{F}$ is a generic rank 1 affine distribution on a 3-manifold X, then every point $\mathbf{x} \in X$ has a neighborhood on which there exist local coordinates (x^1, x^2, x^3) such that

$$\mathcal{F} = \frac{\partial}{\partial x^1} + \operatorname{span}\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right),$$

where $H(\mathbf{x})$ is an arbitrary function on X with $\frac{\partial H}{\partial x^1} \neq 0$.

Point-affine equivalence for affine distributions (C—, Moseley, and Wilkens)

Consider the control-affine system

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{s} \mathbf{f}_i(\mathbf{x}) u^i,$$

The drift vector field $\mathbf{f}_0(\mathbf{x})$ may be replaced by any vector field of the form

$$\mathbf{f}_0(\mathbf{x}) + \sum_{i=1}^{s} \lambda^i(\mathbf{x}) \mathbf{f}_i(\mathbf{x}),$$

and the resulting control system will be affine equivalent to the original system.

But in practice, there is often a preferred choice for the drift vector field, corresponding to a zero value for some physical control inputs. This is particularly true in optimal control, where there is typically a specific control input whose cost function is minimal.

This motivates the following definition:

Definition: A *point-affine distribution* on a manifold X is an affine distribution $\mathcal{F}$ on X, together with a distinguished vector field $\mathbf{f}_0 \in \mathcal{F}$. Two point-affine distributions

$$\mathcal{F}_X = \mathbf{f}_0 + \operatorname{span}\left(\mathbf{f}_1, \ldots, \mathbf{f}_s\right), \qquad \mathcal{F}_Y = \mathbf{g}_0 + \operatorname{span}\left(\mathbf{g}_1, \ldots, \mathbf{g}_s\right)$$

on manifolds X, Y (corresponding to the control-affine systems (9) and (10), respectively) will be called *point-affine equivalent* if there exists a diffeomorphism $\Phi : X \to Y$ such that

$$\Phi_*(\mathcal{F}_1) = \mathcal{F}_2$$

and

$$\Phi_*(\mathbf{f}_0(\mathbf{x})) = \mathbf{g}_0(\Phi(\mathbf{x})).$$

Under point-affine equivalence, functional invariants appear in even lower dimension than for affine equivalence:

Theorem (C— -Moseley-Wilkens).

- *If $\mathfrak{F}$ is a generic rank 1 point-affine distribution on a 2-manifold X, then every point $\mathbf{x} \in X$ has a neighborhood on which there exist local coordinates (x^1, x^2) such that*

$$\mathfrak{F} = x^2 \left(\frac{\partial}{\partial x^1} + J \frac{\partial}{\partial x^2} \right) + span \left(\frac{\partial}{\partial x^2} \right),$$

where $J(\mathbf{x})$ is an arbitrary function on X.

- *If* $\mathfrak{F}$ *is a generic rank 1 point-affine distribution on a 3-manifold* X*, then every point* $\mathbf{x} \in X$ *has a neighborhood on which there exist local coordinates* (x^1, x^2, x^3) *such that*

$$\mathfrak{F} = \left(\frac{\partial}{\partial x^1} + J\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)\right) + span\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right),$$

where $H(\mathbf{x}), J(\mathbf{x})$ *are arbitrary functions on* X *with* $\frac{\partial H}{\partial x^1} \neq 0$.

- *If $\mathfrak{F}$ is a generic rank 2 point-affine distribution on a 3-manifold X, then every point $\mathbf{x} \in X$ has a neighborhood on which there exist local coordinates (x^1, x^2, x^3) such that*

$$\mathfrak{F} = \left((1 + x^3 J_3) \frac{\partial}{\partial x^1} + J_3 \frac{\partial}{\partial x^2} - J_2 \frac{\partial}{\partial x^3} \right) + \mathit{span} \left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right),$$

where $J_2(\mathbf{x}), J_3(\mathbf{x})$ are arbitrary functions on X.

Work in progress: What happens when we put a metric structure akin to sub-Riemannian or sub-Finsler geometry on a point-affine distribution?

Dynamic equivalence for kinematic control systems

Recall that for any smooth curve $\mathbf{x}(t)$ in a manifold X and any non-negative integer k, the *kth prolongation* of $\mathbf{x}(t)$ is the curve

$$p_k(\mathbf{x})(t) = \Big(\mathbf{x}(t), \dot{\mathbf{x}}(t), \ldots, \mathbf{x}^{(k)}(t)\Big)$$

in the jet space $\mathcal{J}^k(X)$.

Two control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{11}$$

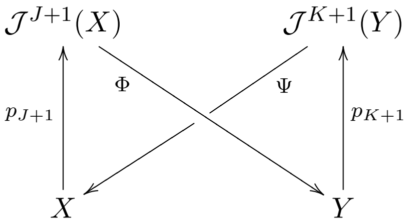
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{v}) \tag{12}$$

on state spaces X, Y, respectively, are called *dynamic equivalent* if there exist integers $J, K \geq -1$ and submersions

$$\Phi : \mathcal{J}^{J+1}(X) \to Y, \qquad \Psi : \mathcal{J}^{K+1}(Y) \to X$$

(defined on appropriate open sets) such that:

- for any solution $\mathbf{x}(t)$ of (11), $(\Phi \circ p_{J+1})(\mathbf{x})(t)$ is a solution to (12),
- for any solution $\mathbf{y}(t)$ of (12), $(\Psi \circ p_{K+1})(\mathbf{y})(t)$ is a solution to (11),
- the following diagram commutes for solutions:



In other words, for any solutions $\mathbf{x}(t)$, $\mathbf{y}(t)$ of (11), (12),

$$\left(\Psi \circ p_{K+1} \circ \Phi \circ p_{J+1}\right)(\mathbf{x}(t)) = \mathbf{x}(t),$$

$$\left(\Phi \circ p_{J+1} \circ \Psi \circ p_{K+1}\right)(\mathbf{y}(t)) = \mathbf{y}(t).$$

These maps are considerably more complicated than those that define a static equivalence; whereas static equivalence is defined by a diffeomorphism

$$\mathbf{y} = \Phi(\mathbf{x}),$$

dynamic equivalence is defined by differential operators:

$$\mathbf{y} = \Phi(\mathbf{x}, \dot{\mathbf{x}}, \dots, \mathbf{x}^{(J+1)})$$
$$\mathbf{x} = \Psi(\mathbf{y}, \dot{\mathbf{y}}, \dots, \mathbf{y}^{(K+1)}).$$

The control systems (11) and (12) define submanifolds

$$\Sigma_1 \subset TX, \qquad \Sigma_2 \subset TY,$$

with local coordinates $(\mathbf{x}, \mathbf{u})$ on Σ_1 and $(\mathbf{y}, \mathbf{v})$ on Σ_2. These, in turn, define prolonged submanifolds

$$\Sigma_1^J \subset \mathcal{J}^{J+1}(X), \qquad \Sigma_2^K \subset \mathcal{J}^{K+1}(Y),$$

with local coordinates

$$(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \ldots, \mathbf{u}^{(J)}), \qquad (\mathbf{y}, \mathbf{v}, \dot{\mathbf{v}}, \ldots, \mathbf{v}^{(K)}),$$

respectively. When restricted to Σ_1^J, Σ_2^K, the maps Φ, Ψ have the form

$$\mathbf{y} = \Phi(\mathbf{x}, \mathbf{u}, \ldots, \mathbf{u}^J),$$
$$\mathbf{x} = \Psi(\mathbf{y}, \mathbf{v}, \ldots, \mathbf{v}^K).$$

Example: Consider the following systems (both with $n = 3,\ s = 2$):

$$\Sigma_1 : \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{y}_1 = v_1 \\ \dot{y}_2 = v_2 \\ \dot{y}_3 = y_2 \end{cases}$$

These systems are not static equivalent, but they are dynamic equivalent (with $J = K = 0$) via the maps

$$(y_1, y_2, y_3) = \Phi(x_1, x_2, x_3, u_1, u_2) = (x_3 - x_1 x_2,\ u_2,\ x_2)$$

$$(x_1, x_2, x_3) = \Psi(y_1, y_2, y_3, v_1, v_2) = \left(-\frac{v_1}{y_2},\ y_3,\ y_1 - \frac{y_3 v_1}{y_2}\right).$$

It is straightforward to show that these maps extend via prolongation to

$$
\begin{aligned}
(y_1, y_2, y_3, v_1, v_2) &= \Phi^{(1)}(x_1, x_2, x_3, u_1, u_2, \dot{u}_1, \dot{u}_2) \\
&= (x_3 - x_1 x_2,\ u_2,\ x_2,\ -x_1 u_2,\ \dot{u}_2)
\end{aligned}
$$

$$
\begin{aligned}
(x_1, x_2, x_3, u_1, u_1) &= \Psi^{(1)}(y_1, y_2, y_3, v_1, v_2, \dot{v}_1, \dot{v}_2) \\
&= \left(-\frac{v_1}{y_2},\ y_3,\ y_1 - \frac{y_3 v_1}{y_2},\ \frac{v_1 v_2 - y_2 \dot{v}_1}{{y_2}^2},\ y_2\right).
\end{aligned}
$$

Check that this is, in fact, a dynamic equivalence between the given systems:

$$\dot{y}_1 = \frac{d}{dt}(x_3 - x_1 x_2)$$

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$$\begin{aligned}\dot{y}_1 &= \frac{d}{dt}(x_3 - x_1 x_2) \\ &= \dot{x}_3 - \dot{x}_1 x_2 - x_1 \dot{x}_2\end{aligned}$$

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$$\begin{aligned}\dot{y}_1 &= \frac{d}{dt}(x_3 - x_1 x_2) \\ &= \dot{x}_3 - \dot{x}_1 x_2 - x_1 \dot{x}_2 \\ &= x_2 u_1 - u_1 x_2 - x_1 u_2\end{aligned}$$

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&= x_2 u_1 - u_1 x_2 - x_1 u_2 \\
&= -x_1 u_2 \\
&= v_1 \qquad \checkmark
\end{aligned}$$

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&= x_2 u_1 - u_1 x_2 - x_1 u_2 \\
&= -x_1 u_2 \\
&= v_1 \qquad \checkmark \\
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&= -x_1 u_2 \\
&= v_1 \qquad \checkmark \\
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&= x_2u_1 - u_1x_2 - x_1u_2 \\
&= -x_1u_2 \\
&= v_1 \qquad \checkmark \\
\dot{y}_2 &= \dot{u}_2 \\
&= v_2 \qquad \checkmark \\
\dot{y}_3 &= \dot{x}_2 \\
&= u_2 \\
&= y_2 \qquad \checkmark
\end{aligned}$$

$$\dot{x}_1 = \frac{d}{dt}\left(-\frac{v_1}{y_2}\right)$$

$$
\begin{aligned}
\dot{x}_1 &= \frac{d}{dt}\left(-\frac{v_1}{y_2}\right) \\
&= \frac{v_1 y_2 - y_3 \dot{v}_1}{{y_3}^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= \frac{d}{dt}\left(-\frac{v_1}{y_2}\right) \\
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$$

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&= y_2 \\
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\end{aligned}
$$

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&= u_1 \qquad \checkmark \\
\dot{x}_2 &= \dot{y}_3 \\
&= y_2 \\
&= u_2 \qquad \checkmark \\
\dot{x}_3 &= \frac{d}{dt}\left(y_1 - \frac{y_3 v_1}{y_2}\right) \\
&= v_1 - \frac{y_2(y_2 v_1 + y_3 \dot{v}_1) - y_3 v_1 v_2}{{y_2}^2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= \frac{d}{dt}\left(-\frac{v_1}{y_2}\right) \\
&= \frac{v_1 y_2 - y_3 \dot{v}_1}{{y_3}^2} \\
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&= y_3 \frac{v_1 v_2 - y_2 \dot{v}_1}{{y_2}^2}
\end{aligned}
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&= y_2 \\
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\dot{x}_3 &= \frac{d}{dt}\left(y_1 - \frac{y_3 v_1}{y_2}\right) \\
&= v_1 - \frac{y_2(y_2 v_1 + y_3 \dot{v}_1) - y_3 v_1 v_2}{{y_2}^2} \\
&= y_3 \frac{v_1 v_2 - y_2 \dot{v}_1}{{y_2}^2} \\
&= x_2 u_1 \quad \checkmark
\end{aligned}
$$

It is straightforward to check that the maps Φ, Ψ act as inverses of each other on solutions.

This example can be constructed by the following method, due to Pomet:

Start with the system

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1.\end{aligned}$$

Construct a *partial prolongation* of this system to a system with 4 states and 2 controls: let

$$z_1 = x_1, \qquad z_2 = x_2, \qquad z_3 = x_3, \qquad z_4 = u_2$$

be the new state variables, and

$$w_1 = u_1, \qquad w_2 = \dot{u}_2$$

the new control variables.

The prolonged system is the control-affine system

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ z_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ z_2 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_2. \tag{13}$$

By the nature of the partial prolongation, the control vector field associated to w_2 is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now perform a *static* transformation among the new state variables in order to arrange that the control vector field associated to w_1,

$$\begin{bmatrix} 1 \\ 0 \\ z_2 \\ 0 \end{bmatrix},$$

is transformed to

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This can be accomplished by the static transformation

$$(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) = (\ z_3 - z_1 z_2,\ \ z_2,\ \ z_4,\ \ z_1),$$

which transforms the system (13) to:

$$\begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \\ \dot{\tilde{z}}_3 \\ \dot{\tilde{z}}_4 \end{bmatrix} = \begin{bmatrix} \dot{z}_3 - z_1 \dot{z}_2 - z_2 \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_4 \\ \dot{z}_1 \end{bmatrix}$$

$$= \begin{bmatrix} -\tilde{z}_3 \tilde{z}_4 \\ \tilde{z}_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w_2.$$

This system is necessarily a partial prolongation of a 3-state system—in this case, the system with state variables

$$\tilde{y}_1 = \tilde{z}_1, \qquad \tilde{y}_2 = \tilde{z}_3, \qquad \tilde{y}_3 = \tilde{z}_2$$

and control variables

$$\tilde{v}_1 = \tilde{z}_4, \qquad \tilde{v}_2 = w_2,$$

given by

$$\begin{aligned} \dot{\tilde{y}}_1 &= -\tilde{y}_2\tilde{v}_1 \\ \dot{\tilde{y}}_2 &= \tilde{v}_2 \\ \dot{\tilde{y}}_3 &= \tilde{y}_2. \end{aligned}$$

Setting

$$y_i = \tilde{y}_i, \qquad i = 1, 2, 3$$
$$v_1 = -\tilde{y}_2 \tilde{v}_1,$$
$$v_2 = \tilde{v}_2$$

transforms this to the system

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \dot{y}_2 &= v_2 \\ \dot{y}_3 &= y_2. \end{aligned}$$

Each step in this transformation process is a dynamic equivalence; therefore the two systems are dynamic equivalent.

The explicit formulas for the maps Φ, Ψ follow directly from this construction.

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Open question: Can every dynamic equivalence be expressed in a similar fashion as a combination of partial prolongations, static equivalences, and deprolongations?

Known results on dynamic equivalence:

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- The number of control variables is an invariant of dynamic equivalence.

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- The number of control variables is an invariant of dynamic equivalence.
- The number of state variables, however, is *not* invariant; for instance, any control system is dynamic equivalent to any prolongation or partial prolongation of itself.
- (Pomet) If Σ_1, Σ_2 are dynamic equivalent systems on state spaces X, Y of dimensions m, n, respectively, then on appropriate open sets:
  - If $m > n$, then for each $\mathbf{x} \in X$, $(\Sigma_1)_{\mathbf{x}}$ is a ruled submanifold of $T_{\mathbf{x}}X$.
  - If $m = n$, then either Σ_1 and Σ_2 are both ruled, or Σ_1 and Σ_2 are static equivalent.

Problem: How to approach the question of equivalence when the maps Φ, Ψ are submersions rather than diffeomorphisms?

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Solution (Stackpole): Pass to the infinite prolongations Σ_1^∞, Σ_2^∞. Then the prolonged maps

$$\Phi^{(\infty)} : \Sigma_1^\infty \to \Sigma_2^\infty, \qquad \Psi^{(\infty)} : \Sigma_2^\infty \to \Sigma_1^\infty$$

become diffeomorphisms, and

$$\Psi^{(\infty)} = \left(\Phi^{(\infty)}\right)^{-1}.$$

It turns out to be most convenient to work on the manifolds

$$\tilde{\Sigma}_i^k = \Sigma_i^k \times \mathbb{R},$$

with coordinate t on the $\mathbb{R}$ factor, for $i = 1, 2$.

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$$\tilde{\Sigma}_i^k = \Sigma_i^k \times \mathbb{R},$$

with coordinate t on the $\mathbb{R}$ factor, for $i = 1, 2$.

For each $k \geq 1$, $\tilde{\Sigma}_1^k$ is a rank s affine distribution on $\tilde{\Sigma}_1^{k-1}$, spanned by the vector fields

$$\left\{ \frac{\partial}{\partial t} + \sum_{j=1}^{n} f_j(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_j} + \sum_{\ell=0}^{k-2} \sum_{r=1}^{s} u_r^{(\ell+1)} \frac{\partial}{\partial u_r^\ell} + \sum_{r=1}^{s} \hat{u}_r^k \frac{\partial}{\partial u_r^{k-1}} \;\middle|\; \hat{u}_1^k, \ldots, \hat{u}_s^k \in \mathbb{R} \right\}.$$

In the limit as $k \to \infty$, $\tilde{\Sigma}_i^\infty$ is similarly equipped with a rank s distribution.

The distributions and the prolongation structures on $\tilde{\Sigma}_1^\infty$, $\tilde{\Sigma}_2^\infty$ are encoded by the following coframings:

$$\bar{\boldsymbol{\omega}} = \begin{pmatrix} \bar{\omega}^{-1} \\ \bar{\boldsymbol{\omega}}^0 \\ \bar{\boldsymbol{\omega}}^1 \\ \bar{\boldsymbol{\omega}}^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} dt \\ d\mathbf{x} - f(\mathbf{x}, \mathbf{u})dt \\ d\mathbf{u} - \dot{\mathbf{u}}dt \\ d\dot{\mathbf{u}} - \ddot{\mathbf{u}}dt \\ \vdots \end{pmatrix}$$

$$\bar{\boldsymbol{\Omega}} = \begin{pmatrix} \bar{\Omega}^{-1} \\ \bar{\boldsymbol{\Omega}}^0 \\ \bar{\boldsymbol{\Omega}}^1 \\ \bar{\boldsymbol{\Omega}}^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} dt \\ d\mathbf{y} - g(\mathbf{y}, \mathbf{v})dt \\ d\mathbf{v} - \dot{\mathbf{v}}dt \\ d\dot{\mathbf{v}} - \ddot{\mathbf{v}}dt \\ \vdots \end{pmatrix}$$

Any other adapted coframing $\boldsymbol{\omega}$ on $\tilde{\Sigma}_1^\infty$ has the form

$$\boldsymbol{\omega} = \mathbf{G}^{-1}\bar{\boldsymbol{\omega}},$$

where $\mathbf{G}$ is an invertible matrix-valued function on $\tilde{\Sigma}_1^\infty$ of the form

$$\mathbf{G} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & \mathbf{0}_{1\times s} & \cdots \\ \mathbf{0}_{n\times 1} & \mathbf{G}_0^0 & \mathbf{0}_{n\times s} & \mathbf{0}_{n\times s} & \mathbf{0}_{n\times s} & \mathbf{0}_{n\times s} & \cdots \\ \mathbf{0}_{s\times 1} & \mathbf{G}_0^1 & \mathbf{G}_1^1 & \mathbf{0}_{s\times s} & \mathbf{0}_{s\times s} & \mathbf{0}_{s\times s} & \cdots \\ \mathbf{0}_{s\times 1} & \mathbf{G}_0^s & \mathbf{G}_1^s & \mathbf{G}_s^s & \mathbf{0}_{s\times s} & \mathbf{0}_{s\times s} & \cdots \\ & & & \vdots & & & \end{bmatrix}.$$

(Adapted coframings $\boldsymbol{\Omega}$ on $\tilde{\Sigma}_2^\infty$ are defined similarly.)

The maps

$$\tilde{\Phi}^{(\infty)} = \Phi^{(\infty)} \times \mathrm{Id}, \qquad \tilde{\Psi}^{(\infty)} = \left(\tilde{\Phi}^{(\infty)}\right)^{-1}$$

define a dynamic equivalence of type (J, K) if and only if for all $k \geq 0$ and any adapted coframings $\boldsymbol{\omega}, \boldsymbol{\Omega}$ on $\tilde{\Sigma}_1^{\infty}$, $\tilde{\Sigma}_2^{\infty}$,

$$\left(\tilde{\Phi}^{(\infty)}\right)^* (\boldsymbol{\Omega}^k) \in \mathrm{span}\{\boldsymbol{\omega}^0, \ldots, \boldsymbol{\omega}^{k+J+1}\},$$
$$\left(\tilde{\Psi}^{(\infty)}\right)^* (\boldsymbol{\omega}^k) \in \mathrm{span}\{\boldsymbol{\Omega}^0, \ldots, \boldsymbol{\Omega}^{k+K+1}\}.$$

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Compare with static equivalence, where the analogous conditions are

$$\left(\tilde{\Phi}^{(\infty)}\right)^* (\boldsymbol{\Omega}^k) \in \mathrm{span}\{\boldsymbol{\omega}^0, \ldots, \boldsymbol{\omega}^{k}\},$$
$$\left(\tilde{\Psi}^{(\infty)}\right)^* (\boldsymbol{\omega}^k) \in \mathrm{span}\{\boldsymbol{\Omega}^0, \ldots, \boldsymbol{\Omega}^{k}\}.$$

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Stackpole analyzes certain aspects of the structure of the linear maps $\left(\tilde{\Phi}^{(\infty)}\right)^*$, $\left(\tilde{\Psi}^{(\infty)}\right)^*$. Using these results, he is able to:

- give a new proof of the known result that when $s = 1$, dynamic equivalence implies static equivalence;
- give a complete categorization for dynamic equivalence with $J = K = 0$ in the case of control-affine systems with $n = 3, \ s = 2$.

This second result is based on Elkin's classification of control-affine systems with $n = 3,\ s = 2$ up to static equivalence. Elkin showed that there are 5 static equivalence classes, exemplified by the systems

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= f(\mathbf{x}, \mathbf{u}), \end{aligned}$$

where $f(\mathbf{x}, \mathbf{u})$ is one of the 5 functions

$$f(\mathbf{x}, \mathbf{u}) = \begin{cases} 0 \\ 1 \\ x_2 \\ x_2 u_1 \\ 1 + x_2 u_1 \end{cases} .$$

Theorem (Stackpole). *The static equivalence classes corresponding to*

$$f = x_2, \qquad f = x_2 u_1, \qquad f = 1 + x_2 u_1$$

are all dynamic equivalent with $J = K = 0$*, while the static equivalence classes corresponding to*

$$f = 0, \qquad f = 1$$

are neither dynamic equivalent to these three nor to each other with $J = K = 0$*.*

While this theorem does not preclude the possibility of additional equivalences at some higher prolongation levels J, K, it represents a significant step towards the ability to distinguish between dynamically inequivalent systems.

Stackpole's main technical result is that when $s = 2$ and $J = K = 0$, the matrix for $\left(\tilde{\Phi}^{(\infty)}\right)^*$ factors as

$$\left(\tilde{\Phi}^{(\infty)}\right)^* = g\mathcal{S}G,$$

where g, G have the form of (possibly time-varying) *static* equivalences and $\mathcal{S}$ is a specific unitary matrix consisting entirely of 0's and 1's.

For example, when $n = 3$:

$$
\mathcal{S} = \left[\begin{array}{c|ccc|cc|cc|cc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
 & & & & & & \vdots & & & & & &
\end{array}\right]
$$

This makes it possible to examine how the invariants of a given system with respect to static equivalence may be transformed by a dynamic equivalence, thereby introducing necessary conditions which must be satisfied for dynamic equivalence.

This result can almost certainly be generalized for larger values of s, J, and K; the only difficulty is that for given values of s, J, K,there will probably be several different possibilities for the matrix $\mathcal{S}$, depending on the ranks of various submatrices of $\left(\tilde{\Phi}^{(\infty)}\right)^*$.

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Open question: Can this idea be extended sufficiently to determine whether there exists a dynamic equivalence of *any* type (J, K) between two given systems?