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Integrability of geometric evolution equations using classical moving frames

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Wave maps and Schrödinger maps are examples of geometric nonlinear evolution equations for a map from space-time into a Riemannian manifold or a Hermitian manifold. In this talk, I will discuss a moving frame formulation of such equations and describe the connection of this formulation to geometric curve and surface flows. As applications, mKdV versions of Schrödinger maps will be obtained, and the bi-Hamiltonian integrability structure of these maps will be derived geometrically in the case when the manifold is a symmetric space or a Lie group.

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Geometric Map Equations

Dynamical variable is a map γ from space-time $\mathbb{R}\times\mathbb{R}$ into a geometrical space M where the equation of motion for γ is invariant under the structure group of M

Examples

1. Vortex filament eqn. $\vec{r}(x,t) \in (\mathbb{R}^3, \wedge) \simeq so(3)$
   $\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xx}\,,\ |\vec{r}_x| = 1$
2. Heisenberg spin model $\vec{S}(x,t) \in S^2 \subset \mathbb{R}^3$
   $\vec{S}_t = \vec{S} \wedge \vec{S}_{xx}\,,\ |\vec{S}| = 1$

related by $\vec{S} = \vec{r}_x$

Intrinsic formulation as Schrodinger map
$\gamma \in S^2 \simeq \mathbb{C}_\infty\,,\quad \gamma_t = J\nabla_x\gamma_x$

3. sigma model $\vec{\phi}(x,t) \in S^3 \subset \mathbb{R}^4$
   $\vec{\phi}_{tx} + (\vec{\phi}_x \cdot \vec{\phi}_t)\vec{\phi} = 0\,,\ |\vec{\phi}| = 1$
4. chiral model $U(x,t) \in SU(2)$
   $(U^{-1}U_t)_x + (U^{-1}U_x)_t = 0\,,\ U^{\dagger} = U^{-1}$

related by Pauli matrices

Intrinsic formulation as Wave map
$\gamma \in S^3,\quad \nabla_x\gamma_t = \nabla_t\gamma_x = 0$

Wave maps and Schrodinger maps are integrable nonlinear PDE systems

Geometric Map Eqn. $\leftrightarrow$ curve flow $\rightarrow$ framing $\downarrow$

Hasimoto variables (invariants modulo gauge transformation)

soldering

soliton eqn. (NLS / SG) $\leftrightarrow$ bi-Hamiltonian structure hierarchy of symmetries + conservation laws

Outline of talk

- review integrability of
  - vortex filament eqn
  - Heisenberg model
  - Schrodinger map into S^2
- present generalizations
  1) vortex filament model $\gamma \in G$ semisimple Lie group
  2) ~~flattening~~ $\Rightarrow$ $\gamma \in \mathfrak{g}$ semisimple Lie algebra

     Heisenberg model $S = \gamma_x$
  3) Schrodinger map $\gamma \in G/H$ Hermitian symmetric space

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what is new?

1. generalization of Hasimoto variables to curve flows in Lie groups & symmetric spaces
2. explicit group-invariant (multicomponent) NLS/SG/mKdV soliton eqns
3. mKdV analog of Schrodinger map
4. frame-independent geometrical formula for bi-Hamiltonian operators and hierarchy of symmetries & conservation laws for curve flows

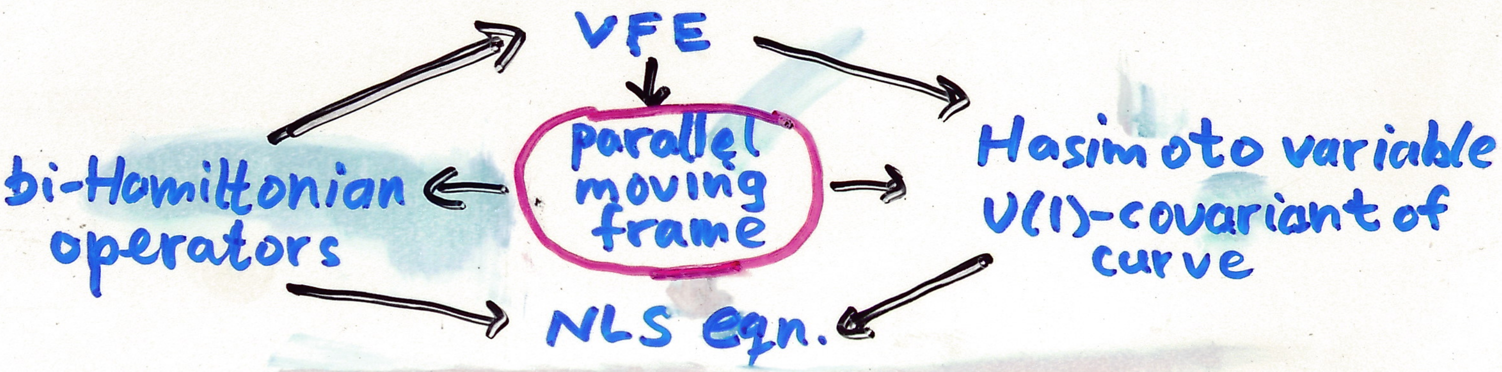
Vortex Filament Equation (VFE)

curve $\vec{r} \in \mathbb{R}^3$

flow $\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xx}, \quad |\vec{r}_x| = 1$

$ds = |\vec{r}_x|\,dx \Rightarrow x = \text{arclength} \Leftrightarrow |\vec{r}_x|_t = 0$

$\Rightarrow$ $\boxed{\text{inelastic (non-stretching) curve flow}}$



Frenet frame $\begin{pmatrix} T \\ N \\ B \end{pmatrix} \Rightarrow \kappa, \tau$ invariants of curve

SO(3) rotation group of gauge transformations acting on frames

$\Rightarrow$ stabilizer of $T = \vec{r}_x$ is SO(2) subgroup acting in normal space of curve

parallel frame $E = \begin{pmatrix} T \\ \cos\theta\, N + \sin\theta\, B \\ -\sin\theta\, N + \cos\theta\, B \end{pmatrix}, \quad \theta_x = -\tau$

frame structure eqn. $E_x = \mathcal{U} E$

$$\mathcal{U} = \begin{pmatrix} 0 & \kappa\cos\theta & -\kappa\sin\theta \\ -\kappa\cos\theta & 0 & 0 \\ \kappa\sin\theta & 0 & 0 \end{pmatrix} \in so(2)_\perp \subset so(3)$$

$\boxed{\text{Cartan matrix belongs to perp space of stabilizer subalgebra in rotation algebra}}$

Hasimoto variable $u = \kappa e^{-i\theta}$

equivalence group of parallel frame

$$E \to R(\phi)E, \quad R(\phi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \in SO(2)$$

$\phi = \text{const.}$

$\Leftrightarrow \quad \theta \to \theta + \phi \quad \Leftrightarrow \quad u \to e^{-i\phi} u$ phase rotation

$e^{-i\phi} \in U(1)$

$\Rightarrow$ $u = \kappa \exp(i\int \tau\, dx)$ is determined from the curve geometrically up to a rigid $U(1)$ equivalence group

$\boxed{\text{Hasimoto variable } u \text{ is } U(1)\text{-covariant of the curve } \vec{r} \in \mathbb{R}^3}$

Cartan equations $E_x = U E, \; E_t = W E$

$$U = \begin{pmatrix} 0 & u_1 & u_2 \\ -u_1 & 0 & 0 \\ -u_2 & 0 & 0 \end{pmatrix}, \quad u = u_1 + i u_2$$

$$W = \begin{pmatrix} 0 & \varpi_1 & \varpi_2 \\ -\varpi_1 & 0 & \varpi_0 \\ -\varpi_2 & -\varpi_0 & 0 \end{pmatrix}, \quad \varpi = \varpi_1 + i\varpi_2 = i u_x$$

$\varpi_0 = \frac{1}{2}|u|^2$ for VFE

zero-curvature $(E_x)_t = (E_t)_x$

$$\Rightarrow \boxed{u_t = D_x \varpi + i u D_x^{-1} \mathrm{Im}(\bar{u}\varpi) \equiv \mathcal{H}(\varpi)}$$

Frame equations $\vec{r}_x = T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T E$

$$\vec{r}_t = h_{\parallel} T + \mathrm{Re}(e^{i\theta} h_\perp) N + \mathrm{Im}(e^{i\theta} h_\perp) B = \begin{pmatrix} h_{\parallel} \\ h_{\perp 1} \\ h_{\perp 2} \end{pmatrix}^T E$$

$h_\perp = h_{\perp 1} + i h_{\perp 2} = i u, \quad h_{\parallel} = 0$ for VFE

zero-torsion $(\vec{r}_x)_t = (\vec{r}_t)_x$

$$\Rightarrow \boxed{\varpi = D_x h_\perp + u D_x^{-1} \mathrm{Re}(\bar{u} h_\perp) \equiv \mathcal{J}(h_\perp)}$$

VFE $\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xx} = \kappa B$

$\Rightarrow h_{\parallel} = 0, \quad h_{\perp} = iu \quad \Rightarrow \quad \varpi = iu_x$

$\Rightarrow \quad u_t = i(u_{xx} + \frac{1}{2}|u|^2 u)$ NLS eqn.

$$\boxed{\vec{r} \text{ satisfies VFE} \iff u = \kappa \exp(i\textstyle\int \tau dx) \text{ satisfies NLS}}$$

Well-known facts:

1) $\mathcal{H}, \mathcal{J}^{-1}$ are compatible Hamiltonian operators

2) $\mathcal{J} = \mathcal{I}^{-1}\mathcal{H}\mathcal{I}^{-1}, \quad \mathcal{I} = -i$

$\mathcal{H}, \mathcal{I}$ are compatible Hamiltonian operators

3) $\mathcal{R} = \mathcal{H}\mathcal{I}^{-1}$ is symmetry recursion operator

$\mathcal{R}^* = \mathcal{I}^{-1}\mathcal{H}$ is conservation-law recursion operator (acting on characteristics)

bi-Hamiltonian structure of NLS

$\varpi = iu_x = \delta H^{(1)}/\delta\bar{u}, \quad H^{(1)} = \mathrm{Im}(\bar{u}_x u)$

$\mathcal{R}^{-1*}(\varpi) = u = \delta H^{(0)}/\delta\bar{u}, \quad H^{(0)} = \frac{1}{2}|u|^2$

$u_t = \mathcal{H}(\delta H^{(1)}/\delta\bar{u}) = \mathcal{E}(\delta H^{(0)}/\delta\bar{u})$

where $\mathcal{H} = D_x + iuD_x^{-1}\mathrm{Im}\,\bar{u}$

$\mathcal{E} = \mathcal{H}\mathcal{I}^{-1}\mathcal{H} = iD_x^2 + i|u|^2 - u_x D_x^{-1}\mathrm{Im}\,\bar{u} + iuD_x^{-1}\mathrm{Re}\,u_x$

compatible Hamiltonian operators

Bi-Hamiltonian structure of VFE

Variational structure w.r.t. u can be soldered onto a variational structure w.r.t. $\vec{r}$

$$H^{(0)} = \tfrac{1}{2}\kappa^2 = \tfrac{1}{2}|\vec{r}_{xx}|^2 \qquad \kappa = N\cdot T_x$$
$$H^{(1)} = -\kappa^2\tau = \vec{r}_{xxx}\cdot(\vec{r}_{xx}\wedge\vec{r}_x) \qquad \tau = B\cdot N_x$$

thm. $\vec{r}_t = \kappa B = \tilde{\mathcal{I}}(\delta H^{(0)}/\delta\vec{r}) = \tilde{\mathcal{E}}(\delta H^{(1)}/\delta\vec{r})$

where $\tilde{\mathcal{I}} = D_x^{-1}(\vec{r}_x\wedge D_x^{-1})$, $\tilde{\mathcal{E}} = \tilde{\mathcal{I}}\,\tilde{\mathcal{H}}^{-1}\tilde{\mathcal{I}}$

$\tilde{\mathcal{H}} = D_x^{-1} + \vec{r}_x D_x^{-1}(\vec{r}_{xx}\cdot D_x^{-1})$

compatible Hamiltonian operators

pf. Let $e^{\parallel} = T = \vec{r}_x$, $e^{\perp} = e^{-i\theta}(N+iB)$

$\Rightarrow$ parallel frame $E = \begin{pmatrix} e^{\parallel} \\ \mathrm{Re}\, e^{\perp} \\ \mathrm{Im}\, e^{\perp} \end{pmatrix}$

structure equations

$e^{\parallel}_x = \mathrm{Re}(\bar{u}e^{\perp})$, $e^{\perp}_x = -u e^{\parallel}$

Take Frechet derivative of structure eqns. and use $e^{\parallel}\cdot\delta e^{\perp} = -e^{\perp}\cdot\delta e^{\parallel}$ $\Rightarrow$

$\delta u = D_x(e^{\perp}\cdot\delta e^{\parallel}) - iuD_x^{-1}\mathrm{Im}(\bar{u}e^{\perp}\cdot\delta e^{\parallel})$, $\delta e^{\parallel} = D_x\delta\vec{r}$

$\Rightarrow$ $\delta H \equiv \mathrm{Re}(\varpi\,\delta\bar{u}) \equiv \mathrm{Re}(\delta\vec{r}\cdot D_x(\mathcal{R}^*(\varpi)e^{\perp}))$

modulo total x-derivatives

Then $\varpi = \delta H/\delta\bar{u} \Rightarrow \delta H/\delta\vec{r} = D_x(\underbrace{\mathcal{R}^*(\delta H/\delta\bar{u})}_{\delta\tilde{H}/\delta\bar{u}}e^{\perp})$

where variations constrained by

$|\vec{r}_x| = 1$ ($\Rightarrow \vec{r}_x\cdot D_x^{-1}\,\delta H/\delta\vec{r} = 0$)

Bi-Hamiltonian structure of Heisenberg model

$\vec{r}$ satisfies VFE $\Longleftrightarrow$ $\vec{S}=\vec{r}_x$ satisfies Heisenberg model

$$\vec{S}_t = (\vec{S}\wedge\vec{S}_x)_x = \vec{S}\wedge\vec{S}_{xx}\ , \quad |\vec{S}|=1$$

thm. $\vec{S}_t = \underset{\sim}{I}(\delta H^{(0)}/\delta\vec{S}) = \underset{\sim}{\mathcal{E}}(\delta H^{(1)}/\delta\vec{S})$

with $H^{(0)} = \frac{1}{2}|\vec{S}_x|^2\ ,\quad H^{(1)} = \vec{S}_{xx}\cdot(\vec{S}_x\wedge\vec{S})$

where $\underset{\sim}{I} = -\vec{S}\wedge\ ,\quad \underset{\sim}{\mathcal{E}} = \underset{\sim}{I}\,\underset{\sim}{H}^{-1}\underset{\sim}{I}$

$\underset{\sim}{H} = D_x + D_x(\vec{S}D_x^{-1}\vec{S}_x\cdot\ )$

compatible Hamiltonian operators

pf. variational relation $\delta\vec{S} = D_x\delta\vec{r}$

$\Rightarrow\ \delta H \equiv \delta\vec{S}\cdot\vec{V} \equiv -\delta\vec{r}\cdot D_x\vec{V}$

modulo total x-derivatives

$\vec{V} = \delta H/\delta\vec{S}\ \Rightarrow\ \delta H/\delta\vec{r} = -D_x\,\delta H/\delta\vec{S}$

(variations constrained by $|\vec{S}|=1 \Rightarrow \vec{S}\cdot\delta\vec{S}=0$)

Geometrical formulation of VFE & Heisenberg model as Schrodinger maps

curve $\vec{r}(x) \in \mathbb{R}^3$, x is arclength $\Leftrightarrow |\vec{r}_x| = 1$

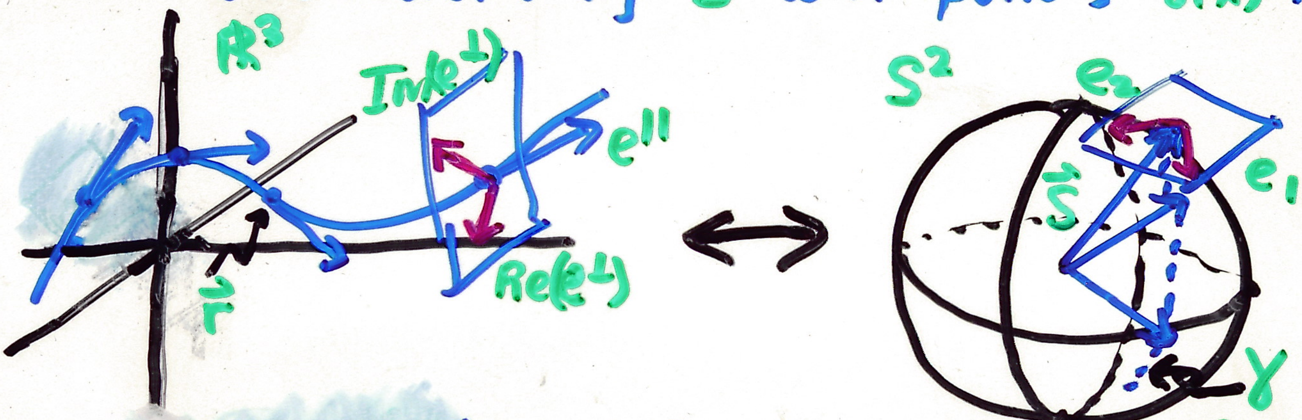
parallel frame $E = \begin{pmatrix} e^{\parallel} \\ Re(e^{\perp}) \\ Im(e^{\perp}) \end{pmatrix}$ along the curve

structure equations $\quad e^{\parallel}_x = Re(\bar{u} e^{\perp})$, $e^{\perp}_x = -u e^{\parallel}$

Parallel-transport $e^{\parallel} = \vec{r}_x$ to the origin in $\mathbb{R}^3$

$\Rightarrow$ identify $e^{\parallel}$ with unit vectors $\vec{S}(x) \in S^2 \subset \mathbb{R}^3$

and identify $\vec{S}$ with points $\gamma(x)$ in S^2



$\Rightarrow$ normal planes of curve $\vec{r}(x)$ in $\mathbb{R}^3$ identified with tangent spaces in S^2 along the curve $\gamma(x)$

$\Rightarrow$ normal vectors $Re(e^{\perp})$ + $Im(e^{\perp}) = Re(-ie^{\perp}) = e^{\parallel} \wedge e^{\perp}$ identified with tangent vectors $e_1, e_2 = * e_1 = J e_1$

$\Rightarrow \quad e^{\perp} = Re(e^{\perp}) + i\, Im(e^{\perp}) \longleftrightarrow e = e_1 + i e_2$

$\Rightarrow$

principal normalvector of curve $e^{\parallel}_x = Re(\bar{u} e^{\perp})$ identified with $\vec{S}_x \longleftrightarrow Re(\bar{u} e) = \gamma_x$ tangent vector of $\gamma(x)$

parallel frame in $\mathbb{R}^3$ satisfies

$$0 = e^\perp_x + u\, e^\parallel = D_x e^\perp + e^\parallel (e^\parallel_x \cdot e^\perp) = D^\perp_x e^\perp$$

where $D^\perp_x = D_x + e^\parallel(e^\parallel_x \cdot\,)$ defines normal derivative w.r.t. $\vec{r}(x)$

$$\left(\vec{v}\cdot e^\parallel = 0 \Rightarrow 0 = D_x(\vec{v}\cdot e^\parallel) = e^\parallel \cdot D_x \vec{v} + e^\parallel_x \cdot \vec{v} = e^\parallel \cdot D^\perp_x \vec{v}\right)$$

$\Rightarrow$ identify $D^\perp_x$ with $D_x + \vec{S}(\vec{S}_x \cdot\,) \leftrightarrow \nabla_x = \gamma_x \lrcorner \nabla$
covariant derivative in S^2 along $\gamma(x)$

$\Rightarrow$ induced frame e in S^2 satisfies $\boxed{\nabla_x e = 0}$

parallel frame $E = \begin{pmatrix} e^\parallel \\ Re(e^\perp) \\ Im(e^\perp) \end{pmatrix}$ along $\vec{r}(x)$ in $\mathbb{R}^3$

$\leftrightarrow$ covariantly-constant frame $\underset{\sim}{E} = \begin{pmatrix} Re(e) \\ Im(e) \end{pmatrix}$ along $\gamma(x)$ in S^2

Identifications: $e^\parallel \leftrightarrow \vec{S} \leftrightarrow \gamma$, $e^\parallel_x \leftrightarrow \vec{S}_x \leftrightarrow \gamma_x$,
$e^\parallel \wedge \leftrightarrow \vec{S}\wedge \leftrightarrow * = J$, $e^\parallel_x \cdot \leftrightarrow \vec{S}_x \cdot \leftrightarrow g(\gamma_x,\ )$,
$D^\perp_x \leftrightarrow D_x + \vec{S}(\vec{S}_x \cdot\,) \leftrightarrow \nabla_x$, $e^\parallel \wedge D^\perp_x \leftrightarrow \vec{S}_x \wedge D_x \leftrightarrow J\nabla_x$

VFE $\vec{r}_t = \kappa B = \vec{r}_x \wedge \vec{r}_{xx}$, $|\vec{r}_x| = 1$ inelastic curveflow

$\leftrightarrow$ Heisenberg model $\vec{S}_t = (\vec{S} \wedge \vec{S}_x)_x = \vec{S} \wedge \vec{S}_{xx}$, $|\vec{S}| = 1$

$\leftrightarrow$ $\gamma_t = J \nabla_x \gamma$ Schrodinger map into S^2
<u>elastic</u> curve flow

arclength along γ is $ds = |\gamma_x|\,dx$ where
$|\gamma_x|^2 = g(\gamma_x, \gamma_x) = |\vec{r}_{xx}|^2 \neq 1$ and $g(\gamma_x,\gamma_x)_t = D_x g(\gamma_x, \gamma_t) \neq 0$

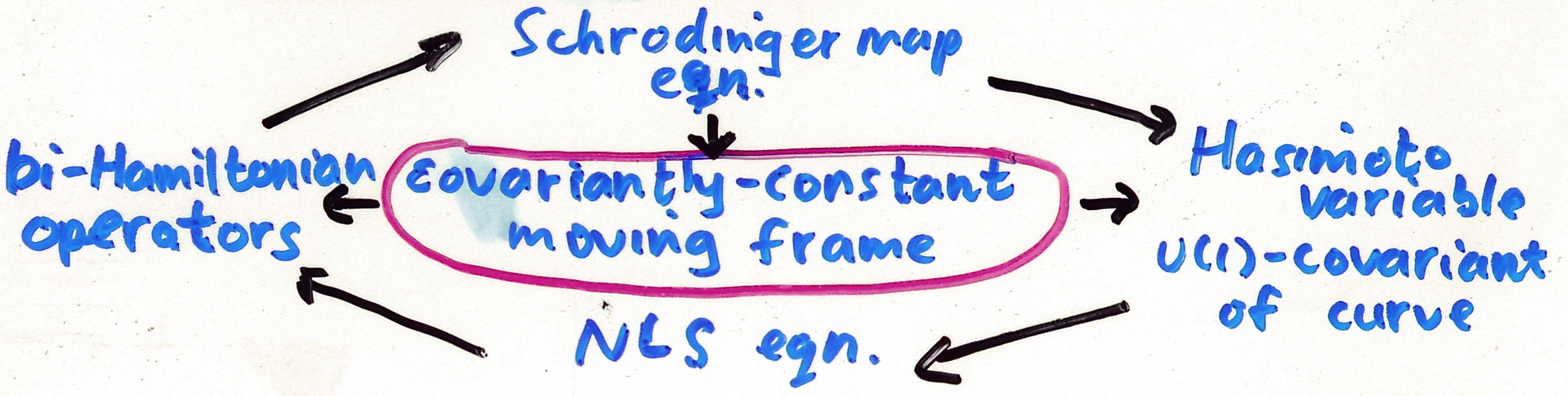
$\Rightarrow$ $|\gamma_x|_t \neq 0$

Bi-Hamiltonian structure of Schrodinger map into S^2

curve $\gamma \in S^2$ arclength $ds = |\gamma_x| dx$

flow $\gamma_t = J \nabla_x \gamma_x \Rightarrow |\gamma_x|_t \neq 0$

$\boxed{\text{elastic curve flow}}$



covariantly-constant frame $\underset{\sim}{E} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $e_2 = Je_1$, $e_1 = -Je_2$

Cartan equations $\nabla_x \underset{\sim}{E} = 0$, $\nabla_t \underset{\sim}{E} = \varpi_0 J \underset{\sim}{E}$

Frame equations $\gamma_x = u_1 e_1 + u_2 e_2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \underset{\sim}{E}$,

$\gamma_t = \varpi_1 e_1 + \varpi_2 e_2 = \begin{pmatrix} \varpi_1 \\ \varpi_2 \end{pmatrix}^T \underset{\sim}{E}$

Use complex formalism

$e = e_1 + ie_2 \Rightarrow Je = -ie$

$u = u_1 + iu_2$ Hasimoto variable

$\varpi = \varpi_1 + i\varpi_2$

$$\Rightarrow \boxed{\begin{array}{l} \nabla_x e = 0, \ \nabla_t e = -i\varpi_0 e \\ \gamma_x = \mathrm{Re}(\bar{u} e), \ \gamma_t = \mathrm{Re}(\bar{\varpi} e) \end{array}}$$

equivalence group of covariantly-constant frame

$e \to \exp(i\phi) e$, $\phi = \text{const.} \Leftrightarrow u \to \exp(i\phi) u$

$\Rightarrow$ u is determined from the parametrized curve up to a rigid U(1) phase rotation

$\boxed{\text{Hasimoto variable } u \text{ is a } U(1)\text{-covariant of the map } \gamma \in S^2}$

∇ is torsion-free $\quad \nabla_t \gamma_x = \nabla_x \gamma_t$

$\Rightarrow \quad u_t = D_x \varpi + iu\varpi_0$

∇ has constant curvature $\quad (\chi = 1)$

$$[\nabla_t, \nabla_x]e = Riem(\gamma_t, \gamma_x)e$$
$$= \chi\left(g(\gamma_x, e)\gamma_t - g(\gamma_t, e)\gamma_x\right)$$

$\Rightarrow \quad D_x \varpi_0 = \chi \,\mathrm{Im}(\bar{u}\varpi)$

Schrodinger map eqn. $\Rightarrow$ $\varpi = iu_x$, $\varpi_0 = \frac{1}{2}|u|^2$
(same as for VFE)

$\Rightarrow$
$$\boxed{\begin{aligned} &u_t = \mathcal{H}(\varpi) = i(u_{xx} + \tfrac{1}{2}|u|^2 u) \quad \text{NLS eqn.} \\ &\mathcal{H} = D_x + iuD_x^{-1}\mathrm{Im}(\bar{u}) \\ &\mathcal{I} = -i \qquad \text{compatible Hamiltonian operators} \end{aligned}}$$

bi-Hamiltonian structure

$\varpi = iu_x = \delta H^{(1)}/\delta\bar{u}$, $H^{(1)} = \mathrm{Im}(\bar{u}_x u)$

$\mathcal{H}^{-1}\mathcal{I}(\varpi) = u = \delta H^{(0)}/\delta\bar{u}$, $H^{(0)} = \frac{1}{2}|u|^2$

$\Rightarrow \quad u_t = \mathcal{H}(\delta H^{(1)}/\delta\bar{u}) = \mathcal{E}(\delta H^{(0)}/\delta\bar{u})$

where $\mathcal{H}, \mathcal{E} = \mathcal{H}\mathcal{I}^{-1}\mathcal{H}$ compatible Hamiltonian operators

Variational structure w.r.t. u can be soldered onto a variational structure w.r.t. γ

$$I = -i \longleftrightarrow \underset{\sim}{I} = -J$$

$$\mathcal{H} = D_x + iu D_x^{-1} \mathrm{Im}(\bar{u}) \longleftrightarrow \underset{\sim}{\mathcal{H}} = \nabla_x + \gamma_x D_x^{-1} g(\gamma_x, \,)$$

correspond geometrically to bi-Hamiltonian operators in Heisenberg model

$$H^{(0)} = \tfrac{1}{2}|u|^2 = \tfrac{1}{2} g(\gamma_x, \gamma_x)$$

$$H^{(1)} = \mathrm{Im}(\bar{u}_x u) = -g(\nabla_x \gamma_x, J\gamma_x)$$

$$H^{(-1)} = g(\xi(\gamma), J\gamma_x), \quad \mathrm{div}_g \xi = 1$$

($\frac{1}{2}\xi$ is homothetic vector field w.r.t. volume form on S^2)

thm.

$$\gamma_t = J\nabla_x \gamma_x = \underset{\sim}{I}(\delta H^{(0)}/\delta\gamma) = \underset{\sim}{\mathcal{E}}(\delta H^{(1)}/\delta\gamma) = \underset{\sim}{\mathcal{H}}(\delta H^{(-1)}/\delta\gamma)$$

where $\underset{\sim}{I}$, $\underset{\sim}{\mathcal{H}}$, $\underset{\sim}{\mathcal{E}} = \underset{\sim}{I}\,\underset{\sim}{\mathcal{H}}^{-1}\underset{\sim}{I}$ compatible Hamiltonian operators

Generalization of VFE

$\vec{r}_t = \vec{r}_x \wedge \vec{r}_{xx}$, $|\vec{r}_x| = 1$ inelastic curve flow

$\vec{r} \in \mathbb{R}^3$

$(\mathbb{R}^3, \wedge) \simeq so(3) \simeq su(2)$

compact semisimple Lie algebra

Cartan subalgebra $\mathfrak{a} \simeq u(1) \subset su(2)$

$i \in \mathfrak{a} \Leftrightarrow J = ad(i)$ satisfies

$\underbrace{J^2 = -id \text{ in } \mathfrak{a}_\perp \subset su(2)}_{\text{complex structure}}$

$\gamma \in G$ compact semisimple Lie group with complex structure

$\gamma_t = [\gamma_x, \nabla_x \gamma_x]$, $g(\gamma_x, \gamma_x) = 1$

inelastic curve flow

VFE in Lie group

parallel frame

bi-Hamiltonian operators

Hasimoto variable

group-invariant NLS eqn. in Lie algebra

How to generalize a parallel frame from $\mathbb{R}^3$ to G?

Frame bundle structure of Lie groups

G = compact semisimple Lie group

$\mathfrak{g}$ = Lie algebra $\langle\, , \rangle$ = Killing form (negative-definite)

$K = G \times G$, $\operatorname{diag} K \simeq G$

$\Rightarrow G \simeq K/\operatorname{diag} K = M$ symmetric space

view K as principal G-bundle over M

Ω = Maurer-Cartan form on K
flat connection 1-form

Fix a local section of K

$\psi : \mathcal{V} \subset M \to K \simeq \mathcal{V} \times G$ local trivialization

Ω_ψ = pull-back of Ω to $\mathcal{V}$ by ψ
$\mathfrak{k}$-valued 1-form on $G \simeq M$

$\Rightarrow d\Omega_\psi + \frac{1}{2}[\Omega_\psi, \Omega_\psi] = 0$ zero-curvature equation

write $\Omega_\psi = (e, \omega)$

$e : T_pG \to \mathfrak{g}$ linear coframe at point $p \in G$

$\omega : T_pG \to \mathfrak{g}$ linear connection

$\Rightarrow$ $\boxed{de + [\omega, e] = 0,\ d\omega + \frac{1}{2}[\omega,\omega] = -\frac{1}{2}[e,e]}$

frame structure equations

Relationship to classical frame

$\{g_{(l)}\}$ = any fixed o.n. basis of $\mathfrak{g}$

$C^l{}_{pq}$ = structure constants of $[\,,\,]$

$(l,p,q = 1,\dots,\dim\mathfrak{g})$

define $e^l = \langle e, g_{(l)}\rangle$, $\omega^l = \langle \omega, g_{(l)}\rangle$

$\Rightarrow$ frame covectors $\{e^l\}$ (dual frame)

connection 1-forms $\{\omega^l\}$

$$de^l = C^l{}_{pq}\, e^p \wedge \omega^q$$

Gauge transformations

$\tilde{\psi} = \psi f$ change of local section

$f : U \subset G \to G$ smooth function

$\Rightarrow$ $\tilde{e} = Ad(f)e$, $\tilde{\omega} = Ad(f)\omega + f\,df^{-1}$

Soldering relations

$X \lrcorner e = e_X$, $X \lrcorner \omega = \omega_X$ for vectors $X \in T_pG$

$g(X,Y) = -\langle e_X, e_Y\rangle \Rightarrow$ metric tensor g on G

$[X,Y,Z] \lrcorner e = [[e_X, e_Y], e_Z] \Rightarrow$ Lie triple-product on G

$$[X,Y,Z] + [Y,X,Z] = 0 = [X,Y,Z] + \text{cyclic}$$

$\nabla_X e = [e, \omega_X] \Rightarrow$ covariant derivative ∇ on G

zero torsion $\underline{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

covariantly-constant curvature $\underline{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$
$= -[X,Y,Z]$

Inelastic curve flows in Lie groups

$\gamma(x,t) \in G$ compact semisimple

dx = arclength $\Longleftrightarrow$ $g(\gamma_x, \gamma_x) = 1$

e, ω $\mathfrak{g}$-valued linear coframe & connection

Write $\gamma_x \rfloor e = e_\parallel$, $\gamma_x \rfloor \omega = u$
$\gamma_t \rfloor e = h$, $\gamma_t \rfloor \omega = \varpi$

Make gauge choice

(1) $D_x e_\parallel = D_t e_\parallel = 0$ via local (x-dependent) gauge transformation $Ad(G)$

$\Rightarrow$ $e_\parallel$ = const. , $-\langle e_\parallel, e_\parallel \rangle = 1$

$\Rightarrow$ decomposition $\mathfrak{g} = \mathfrak{g}_\parallel \oplus \mathfrak{g}_\perp$
given by $ad(e_\parallel)\mathfrak{g}_\parallel = 0$, $\langle \mathfrak{g}_\perp, \mathfrak{g}_\parallel \rangle = 0$

$\Rightarrow$ algebraic structure
$[\mathfrak{g}_\parallel, \mathfrak{g}_\parallel] \subseteq \mathfrak{g}_\parallel$, $[\mathfrak{g}_\parallel, \mathfrak{g}_\perp] \subseteq \mathfrak{g}_\perp$
$ad(e_\parallel)$ invertible in $\mathfrak{g}_\perp$

(2) $\langle u, \mathfrak{g}_\parallel \rangle = 0$ via local (x-dependent) gauge transformation $Ad(G)$ restricted to preserve $e_\parallel$

$\Rightarrow$ $u \in \mathfrak{g}_\perp$ (natural generalization of Euclidean parallel frame)

$\boxed{\text{(1) \& (2) called } \textbf{G-parallel frame}}$

w.l.g. assume $e_\parallel \in \mathfrak{a}$ Cartan subalgebra of $\mathfrak{g}$
(via rigid transformation $Ad(G)$)

Equivalence group of G-parallel frame

$G_{\parallel} \subset G$ subgroup defined by rigid (x-independent) gauge transformations

$f : \gamma \in G \to G_{\parallel}$ s.t. $D_x f = 0$, $\mathrm{Ad}(f) e_{\parallel} = e_{\parallel}$

$\Rightarrow$ $u \in \mathfrak{g}_{\perp}$ is a covariant of the curve γ relative to the group $G_{\parallel}$
(γ determines u geometrically up to rigid action of $G_{\parallel}$)

$\Rightarrow$ natural generalization of Hasimoto variable

Frame structure equations

pull back $0 = d\Omega_\gamma + \frac{1}{2}[\Omega_\gamma, \Omega_\gamma]$
from G to 2-surface $\gamma(x,t) \in G$

$$\left.\begin{aligned} &\mathrm{ad}(e_{\parallel})\varpi^{\perp} + D_x h_{\perp} + [u, h_{\parallel}] + [u, h_{\perp}]_{\perp} = 0 \\ &D_x h_{\parallel} + [u, h_{\perp}]_{\parallel} = 0 \end{aligned}\right\} \text{zero torsion}$$

$$\left.\begin{aligned} &D_t u - D_x \varpi^{\perp} + [\varpi^{\parallel}, u] + [\varpi^{\perp}, u]_{\perp} = \mathrm{ad}(e_{\parallel}) h_{\perp} \\ &D_x \varpi^{\parallel} + [u, \varpi^{\perp}]_{\parallel} = 0 \end{aligned}\right\} \text{covariantly-const. curvature}$$

where $h = h_{\parallel} + h_{\perp}$, $\varpi = \varpi^{\parallel} + \varpi^{\perp}$
w.r.t. $\mathfrak{g} = \mathfrak{g}_{\parallel} \oplus \mathfrak{g}_{\perp}$

Bi-Hamiltonian operators

$u, \varpi^\perp, h^\perp = \mathrm{ad}(e_{11})h_\perp \in \mathfrak{g}_\perp$

$$\Rightarrow \boxed{u_t = \mathcal{H}(\varpi^\perp) + h^\perp, \quad \varpi^\perp = \mathcal{J}(h^\perp)}$$

$$\mathcal{H} = D_x + [u,\cdot]_\perp - [u, D_x^{-1}[u,\cdot]_{\parallel}]$$

$$\mathcal{J} = -\mathcal{I}^{-1}\mathcal{H}\mathcal{I}^{-1}, \quad \mathcal{I} = \mathrm{ad}(e_{11})$$

thm. $\mathcal{H}, \mathcal{I}$ compatible Hamiltonian operators $\Rightarrow$ $\mathcal{R} = \mathcal{H}\mathcal{I}^{-1}$ symmetry recursion operator

$\mathcal{R}^* = \mathcal{I}^{-1}\mathcal{H}$ conservation-law recursion operator (acting on characteristics)

Holds for arbitrary inelastic curve flows

NLS flow

$h^\perp = \mathrm{ad}(e_\parallel)u \Rightarrow \varpi^\perp = -\mathrm{ad}(e_\parallel)^{-1}u_x$

$(\Rightarrow h_\parallel = 0,\ \varpi^\parallel = [\mathrm{ad}(e_\parallel)^{-1}u_x, u])$

$$u_t - \mathrm{ad}(e_\parallel)u = -\mathrm{ad}(e_\parallel)^{-1}u_{xx} - [u, \mathrm{ad}(e_\parallel)^{-1}u_x]_\perp + \tfrac{1}{2}[u,[u,\mathrm{ad}(e_\parallel)^{-1}u]_\parallel]$$

$$= \mathcal{H}(\delta H^{(1)}/\delta u) = \mathcal{E}(\delta H^{(0)}/\delta u)$$

$H^{(0)} = \frac{1}{2}\langle u,u\rangle$, $H^{(1)} = -\frac{1}{2}\langle \mathrm{ad}(e_\parallel)^{-1}u_x, u\rangle$

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suppose $\mathfrak{g}_\perp \subseteq \mathfrak{a}_\perp$ has complex structure J

$\Leftrightarrow J^2 = -\mathrm{id}$ in $\mathfrak{g}_\perp$

choose $e_\parallel \in \mathfrak{a}$ s.t. $\mathrm{ad}(e_\parallel) = \sqrt{\chi}\, J$

$\Leftrightarrow \mathrm{ad}(e_\parallel)^2 = -\chi\, \mathrm{id}$ in $\mathfrak{g}_\perp$

$\Rightarrow u_t - \sqrt{\chi}\, J u = -\frac{1}{\sqrt{\chi}}\left(J u_{xx} - \frac{1}{2}\mathrm{ad}(u)^2 J u\right)$

$G_\parallel$-invariant NLS eqn.

Variational structure w.r.t. u can be soldered onto a variational structure w.r.t. γ

$-\mathrm{ad}(e_\parallel)u = e \lrcorner \nabla_x \gamma_x$, $[e\lrcorner X, e\lrcorner Y] = e\lrcorner\, \mathrm{ad}(X)Y$

$e\lrcorner \gamma_t = h_\parallel + h_\perp = u$ for NLS flow

$\Rightarrow$ **thm.** $\gamma_t = \frac{1}{\chi}\mathrm{ad}(\gamma_x)\nabla_x\gamma_x = \mathcal{H}(\delta H^{(1)}/\delta\gamma) = \mathcal{E}(\delta H^{(0)}/\delta\gamma)$

where $\mathcal{E} = \mathcal{H}\mathcal{I}^{-1}\mathcal{H}$, $\mathcal{H} = \nabla_x^{-1} + \frac{1}{\chi}[\nabla_x^{-1}, \nabla_x\gamma_x, \gamma_x]$,

$\mathcal{I} = \nabla_x^{-1}\mathrm{ad}(\gamma_x)\nabla_x^{-1}$ compatible Hamiltonian ops.

$H^{(0)} = \frac{1}{2} g(\nabla_x\gamma_x, \nabla_x\gamma_x)$, $H^{(1)} = \frac{1}{2\chi} g(\nabla_x^2\gamma_x, \mathrm{ad}(\gamma_x)\nabla_x\gamma_x)$

mKdV flow

$h^\perp = u_x$

$\Rightarrow u_t - u_x = u_{xxx} + \frac{3}{2\chi}\,\mathrm{ad}([Ju,u])u_x)$

G_{11}-invariant bi-Hamiltonian mKdV eqn.

$\Rightarrow \gamma_t = \nabla_x{}^2\gamma_x + \frac{3}{2\chi}\,\mathrm{ad}(\nabla_x\gamma_x)^2\gamma_x$ mKdV map into G

SG flow

$\varpi^\perp = \mathcal{J}(h^\perp) = 0 \Rightarrow u_t = h^\perp \Rightarrow \mathcal{J}(u_t) = 0$

$\Rightarrow u_{tx} = \frac{1}{\chi}\,\mathrm{ad}(Ju)D_x^{-1}(\mathrm{ad}(Ju)u_t)$

G_{11}-invariant sine-Gordon eqn.

$\Rightarrow \nabla_x\gamma_t = 0$ wave map into G (chiral model)

Heisenberg model in Lie algebras

$\gamma \in \mathfrak{g}$ compact semisimple Lie algebra

$\gamma_t = [\gamma_x, \gamma_{xx}]$, $-\langle \gamma_x, \gamma_x \rangle = 1$

VFE in $\mathfrak{g}$ inelastic curve flow

$u = -\mathrm{ad}(e_{11})^{-1} e \rfloor D_x \gamma_x$ $\mathfrak{g}_\perp$-valued

Hasimoto variable defined by parallel frame along curve γ

$\Rightarrow$ $u_t = \mathcal{H}(\varpi^\perp)$, $\varpi = \mathcal{J}(h^\perp)$

where $\mathcal{J} = -\mathcal{I}^{-1} \mathcal{H} \mathcal{I}^{2^{-1}}$, $\mathcal{H}$, $\mathcal{I}$ same operators as for Lie groups

Suppose $\mathrm{ad}(e_{11})^2 = -\chi\, \mathrm{id}$ in $\mathfrak{g}_\perp$

$\Rightarrow$ $J = \sqrt{\chi}\, \mathrm{ad}(e_{11})$ complex structure

$h^\perp = Ju$ generates NLS eqn. on u (with no phase term $\Leftrightarrow$ $\mathfrak{g}$ is flat space)

Identify γ_x with unit vector S at the origin in $\mathfrak{g}$

VFE $\Rightarrow$ $\boxed{S_t = [S, S_{xx}], \quad -\langle S, S \rangle = 1 \quad \text{Heisenberg model in } \mathfrak{g}}$

bi-Hamiltonian structure soldered from $\mathcal{H}$ & $\mathcal{I}$ $\Rightarrow$

$$\boxed{\tilde{\mathcal{I}} = -\mathrm{ad}(S), \quad \tilde{\mathcal{H}} = D_x^\perp + \mathrm{ad}(S_x) D_x^{-1} \mathrm{ad}(S_x)}$$

Schrodinger map in symmetric spaces

$\gamma \in M = G/H$ Hermitian symmetric space

$\gamma_t = J \nabla_x \gamma_x$, $g(\gamma_x, \gamma_x)_t = D_x g(\gamma_x, \gamma_t) \neq 0$

elastic curve flow

$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ symmetric Lie algebra

$\Rightarrow$ $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ Lie subalgebra

$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$

Maurer-Cartan form on G yields

linear coframe $e : T_pM \to \mathfrak{m}$

linear connection $\omega : T_pM \to \mathfrak{h}$

gauge group is $H \subset G$

soldering $e \rfloor g(X, Y) = -\langle e \rfloor X, e \rfloor Y \rangle$

$e \rfloor JX = [i, e \rfloor X] = \mathrm{ad}(i)(e \rfloor X)$

for all vectors $X, Y \in T_pM$

$i \in \mathfrak{a} \cap \mathfrak{m}$ Cartan subspace

s.t. $\begin{cases} \mathrm{ad}(i)^2 = -\mathrm{id} \text{ in } \mathfrak{m} \Rightarrow \text{complex structure} \\ \mathrm{ad}(i)\mathfrak{h} = 0 \Rightarrow \mathfrak{h} \text{ is centralizer} \end{cases}$

Choose covariantly-constant frame

$\nabla_x e = 0 \Rightarrow \omega \rfloor \gamma_x = 0$

write $e \rfloor \gamma_x = u \in \mathfrak{m}$, $e \rfloor \gamma_t = \varpi \in \mathfrak{m}$

$\omega \rfloor \gamma_t = h \in \mathfrak{h}$

zero-curvature Cartan equation

$\Rightarrow$ frame structure equations

$$\boxed{\begin{array}{l} u_t = \mathcal{H}(\varpi) \\ \mathcal{H} = D_x - ad(u) D_x^{-1} ad(u) \,,\ \mathcal{I} = ad(i) \\ \text{compatible Hamiltonian operators} \end{array}}$$

$h = -D_x^{-1}[u, \varpi]$ determines curve flow

Schrodinger map eqn. $\Rightarrow$ $\varpi = ad(i)u_x$

$h = \frac{1}{2}[ad(i)u, u]$

$$u_t = ad(i)\, u_{xx} - \frac{1}{2}\, ad(u)^2 ad(i) u$$

H-invariant bi-Hamiltonian NLS eqn.

Soldering yields bi-Hamiltonian structure for Schrodinger map eqn.

$$\underset{\sim}{\mathcal{I}} = J \,,\ \underset{\sim}{\mathcal{H}} = \nabla_x + ad(\gamma_x) \nabla_x^{-1} ad(\gamma_x)$$