Submanifolds, moving frames, and integrable systems

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Outline of lecture:

- Isothermic surfaces in $\mathbb{R}^3$, classical theory
- U/K-system
- Isothermic hypersurfaces in $\mathbb{R}^{n+1}$

Isothermic surfaces in $\mathbb{R}^3$

An immersion $f(x_1, x_2) \in \mathbb{R}^3$ is called isothermic if (x_1, x_2) is a conformal line of curvature coordinate system, i.e.,

$$\mathrm{I} = e^{2u}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \quad \mathrm{II} = \mathrm{e}^{\mathrm{u}}(\mathrm{r}_1\,\mathrm{d}x_1^2 + \mathrm{r}_2\,\mathrm{d}x_2^2)$$

for some function u, r_1, r_2.

The Gauss-Codazzi equation (GCE) is

$$\begin{cases} u_{x_1x_1} + u_{x_2x_2} + r_1 r_2 = 0, \\ (r_1)_{x_2} = u_{x_2} r_2, \\ (r_2)_{x_1} = u_{x_1} r_1. \end{cases}$$

Examples

- Constant mean curvature surfaces in $\mathbb{R}^3$ away from umbilic points admit isothermic coordinates
- a local conformal parametrization of S^2, i.e., composition of analytic functions with the inverse of the Sterographic projection.

Christoffel pair

Note that if (u, r_1, r_2) is a solution of the GCE, then so is $(-u, r_1, -r_2)$. What are the relation between the corresponding isothermic surfaces?

Thm. (Christoffel) M_1, M_2 surfaces in $\mathbb{R}^3$, $\phi : M_1 \to M_2$ a conformal, orientation reversing diffeomorphism, T_pM_1 is parallel to $T_{\phi(p)}M_2$ for all $p \in M_1$. Then both M_1 and M_2 are isothermic such that with (u, r_1, r_2) and $(-u, r_1, -r_2)$ the corresponding solutions of the GCE resp. We call the pair of isothermic immersions (f_1, f_2) a Christoffel pair.

Ribaucour transform

$M, \tilde{M}$ hypersurfaces in $\mathbb{R}^{n+1}$, a diffeo $\phi : M \to \tilde{M}$ is a Ribaucour transform if

- the normal line of M at p and the normal line of $\tilde{M}$ at $\phi(p)$ meets at equi-distance $r(p)$ for each $p \in M$, i.e., $M, \tilde{M}$ are envelopes of a n parameter hypersphere congruence,
- ϕ maps principal directions of M to those of $\tilde{M}$.

Thm. (Darboux) Given an isothermic surface M in $\mathbb{R}^3$, there is a one parameter family of Ribaucour transforms from M obtained by solving a compatible system of first order PDE system of five functions.

Lax pair for the GCE of isothermic surfaces

Cieslinski-Goldstein-Sym (1995): The GCE for isothermic surfaces in $\mathbb{R}^3$ has a Lax pair, i.e., it is given by the flatness of the $o(4,1)$-valued connections on $\mathbb{R}^2$:

$$\theta_\lambda = \begin{pmatrix} \tau_1 & \delta\lambda \\ \delta^\sharp\lambda & \tau_2 \end{pmatrix}, \qquad \text{where}$$

$$\tau_1 = (w_{AB}) = \begin{pmatrix} 0 & u_{x_2}\,\mathrm{dx}_1 - \mathrm{u}_{\mathrm{x}_1}\,\mathrm{dx}_2 & r_1\,\mathrm{dx}_1 \\ & 0 & r_2\,\mathrm{dx}_2 \\ & & 0 \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} 0 & \mathrm{du} \\ \mathrm{du} & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} 0 & \mathrm{dx}_1 \\ \mathrm{dx}_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \delta^\sharp = -J\delta^t.$$

Remark. This is the $\frac{O(4,1)}{O(3)\times O(1,1)}$-system, the first order system associated to the rank 2 symmetric space $\frac{O(4,1)}{O(3)\times O(1,1)}$.

Symmetric spaces and Cartan decompostion

Let U/K be a (pseudo-) Riemannian symmetric space defined by involution σ,

$$\mathcal{U} = \mathcal{K} + \mathcal{P}$$

a Cartan decomposition, (i.e., the eigenspace decomposition of σ_* with eigenvalues $1, -1$) $\mathcal{A}$ a maximal abelian subalgebra in $\mathcal{P}$, and $\mathcal{A}^\perp$ the orthogonal complement of $\mathcal{A}$ in $\mathcal{P}$ w.r.t. the Killing form. The dim of $\mathcal{A}$ is the rank of $\frac{U}{K}$.

$\frac{U}{K}$-system

The $\frac{U}{K}$-system (T-) is the following system for $v : \mathbb{R}^n \to \mathcal{A}^\perp$:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] + [[a_i, v], [a_j, v]] = 0, \quad 1 \leq i \neq j \neq n.$$

Or equivalently,

$$\theta_\lambda = \sum_{i=1}^{n} (a_i\lambda + [a_i, v]) \, \mathrm{dx_i}$$

is a flat $\mathcal{U}_\mathbb{C}$-valued connection 1-form on $\mathbb{R}^n$ for all complex parameter λ.

Given a solution of the $\frac{U}{K}$-system, there is a unique $E(x, \lambda)$ satisfying

$$E^{-1} \, \mathrm{dE} = \theta_\lambda, \quad \mathrm{E}(0, \lambda) = \mathrm{I}.$$

Such E is called the frame of the solution v.

Curved flats and Flat Abelian submanifolds

- Ferus-Pedit: An n-dim submanifold M in a rank n symmetric space $\frac{U}{K}$ is a curved flat if M is tangent to a totally geodesic n dim flat n-submanifold of $\frac{U}{K}$ at p for each $p \in M$.
- T-: M^n in $\mathcal{P}$ is a flat, abelian submanifold if the induced metric is flat, T_pM is a maximal abelian subalgebra in $\mathcal{P}$ for each $p \in M$, and the normal bundle of M is flat.

Equivalence

The $\frac{U}{K}$-system is equivalent to both the GCE for curved flats in $\frac{U}{K}$ and the GCE for flat abelian submanifolds in $\mathcal{P}$.

Thm. if $E(x,\lambda)$ is the frame for a solution v of the $\frac{U}{K}$-system, then

1. $\eta = \frac{\partial E}{\partial \lambda} E^{-1} \big|_{\lambda=0}$ is a flat, abelian submanifold in $\mathcal{P}$,
2. $f(x) = E(x,1)E(x,-1)^{-1}$ is a curved flat in $\frac{U}{K}$,
3. if $\frac{U}{K} = \frac{O(4,1)}{O(3)\times O(1,1)}$, then η in (1) is of the form $\eta = \begin{pmatrix} 0 & Y \\ Y^\sharp & 0 \end{pmatrix}$, where Y is 3×2 valued and $Y = (f_1, f_2)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a Christoffel pair.

Dressing action of the Rational loop group

Let $\hat{G}$ denote the group of rational maps $g : S^2 \to U_{\mathbb{C}}$ satisfying the $\frac{U}{K}$-reality condition

$$\overline{g(\bar{\lambda}))} = g(\lambda), \quad g(-\lambda) = \sigma(g(\lambda))$$

and $g(\infty) = \mathrm{I}$.

Uhlenbeck- T: Let $E(x, \lambda)$ be the frame of a solution v of the $\frac{U}{K}$-system, and $g \in \hat{G}$. Then there is an open subset $\mathcal{O}$ of 0 in $\mathbb{R}^n$ such that for each $x \in \mathcal{O}$ we can factor

$$g^{-1}(\lambda)E(x, \lambda) = \tilde{E}(x, \lambda)\tilde{g}(x, \lambda)$$

such that $\tilde{g}(x, \cdot) \in \hat{G}$ and $E(x, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$.

Moreover,

- $\tilde{E}$ is again a solution $\tilde{v}$ of the $\frac{U}{K}$-system,
- $\tilde{E}$, so is $\tilde{v}$, can be obtained explicitly and algebraically via residue calculus.
- $(g, v) \mapsto \tilde{v}$ defines an action of $\hat{G}$ on the space of local solutions of the $\frac{U}{K}$-system,
- $\tilde{g}$ can also be obtained by solving a system of non-linear first order PDE (Bäcklund, Darboux transf),
- the Bianchi permutability formula follows from the relation in $\hat{G}$,
- if we apply this to $\frac{O(4,1)}{O(3)\times O(1,1)}$-system for $g \in \hat{G}$ with two simple poles, then we get the classical Darboux Theorem for Ribaucour transf for isothermic surfaces.

Isothermic hypersurfaces in $\mathbb{R}^{n+1}$

Joint work with Neil Donaldson:

- An orthogonal coordinate system x of (M, ds^2) is called C-coordinate system if $\mathrm{ds}^2 = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{g}_{\mathrm{ii}}\, \mathrm{dx}_{\mathrm{i}}^2$ satisfying

$$g_{11} + \ldots + g_{n-1,n-1} - g_{nn} = 0.$$

- An immersion $f(x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ is isothermic if x is both a C-coordinate and a line of curvature coordinate system.
- A diffeo $\phi : M \to \tilde{M}$ between two hypersurfaces is a Combescure transf if T_pM is parallel to $T_{\phi(p)}\tilde{M}$ for each $p \in M$.

Christoffel n-tuple

Let $\mathcal{O}$ be an open subset in $\mathbb{R}^n$, and $\mathcal{M}_{n+1,n}$ the space of $(n+1)\times n$ matrices. A map $(f_1,\ldots,f_n):\mathcal{O}\to\mathcal{M}_{n+1,n}$ is called a Christoffel n-tuple if

- $f_1(x)$ is parametrized by C-coordinates,
- $f_i(x)\mapsto f_{i+1}(x)$ is a conformal Combescure transf and preseves C-coordinates for each $1\le i\le n-1$,
- $\mathrm{df}_1,\ldots,\mathrm{df}_n$ are linearly independent,

Thm. (D-T)

1. If $(f_1, \ldots, f_n)$ is a Christoffel n-tuple, then each f_i is isothermic.
2. $(f_1, \ldots, f_n)$ gives rise to a solution of the $\frac{O(2n,1)}{O(n+1)\times O(n-1,1)}$-system.
3. Conversely, we can associate to each solution of the $\frac{O(2n,1)}{O(n+1)\times O(n-1,1)}$-system and a null basis of $\mathbb{R}^{n-1,1}$ a Christoffel n-tuples.
4. The dressing action of $g \in \hat{G}$ with two simple poles gives rise to Darboux transf for Christoffel n-tuples.