Geometry of hyperbolic conservative systems

Irina Kogan[1] joint work with Kris Jenssen[2]

[1]North Carolina State University

[2]Penn State University

June 13, 2011

The talk is based on:

1. Jenssen, H. K., Kogan, I. A., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010) pp. 211-254.
2. Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws, (2011) `http://arxiv.org/abs/1103.5250`

Problem 1: Jacobians with prescribed eigenfields

Given: (i) An affine coordinates $u = (u^1, \dots, u^n)$ $(\nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_j} = 0)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

(ii) A frame $\mathcal{R} = (r_1, \dots, r_n)$ on Ω.

Find all: vector value maps $(f^1, \dots, f^n) \colon \Omega \to \mathbb{R}^n$, s. t. $r_1|_{\bar{u}}, \dots, r_n|_{\bar{u}}$ are right eigenvectors of the Jacobian matrix $D_u f(\bar{u})$, $\forall \bar{u} \in \Omega$;

Equivalently, find all maps $(\lambda^1, \dots, \lambda^n) \colon \Omega \to \mathbb{R}^n$, s.t.

$$\boxed{J(u) = R(u)\,\Lambda(u)\,L(u)} \text{ is a Jacobian matrix,}$$

where matrix $R = (R_i^j)$ is defined by $r_i = \sum_{i=1}^{n} R_i^j(u) \frac{\partial}{\partial u^j}$,

$L := R^{-1}$ and $\Lambda(u) := \mathrm{diag}[\lambda^1(u), \dots, \lambda^n(u)]$

Problem 2: Hessian inner-products with prescribed orthogonal frame

Given: (i) An affine coordinates $u = (u^1, \ldots, u^n)$ $(\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = 0)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

(ii) A frame $\mathcal{R} = (r_1, \ldots, r_n)$ on Ω.

Find all: functions $\eta \colon \Omega \to \mathbb{R}$, s. t. frame $\mathcal{R}$ is orthogonal with respect to the inner product defined by the Hessian matrix $D_u^2 \eta$.

Equivalently, find all maps $(\beta^1, \ldots, \beta^n) \colon \Omega \to \mathbb{R}^n$, s.t.

$$\boxed{H(u) = L^T(u)\, \mathcal{B}(u)\, L(u)} \text{ is a Jacobian matrix,}$$

where matrix $R = (R_i^j)$ is defined by $r_i = \sum_{i=1}^{n} R_i^j(u) \frac{\partial}{\partial u^j}$,

$L := R^{-1}$ and $\mathcal{B}(u) := \mathrm{diag}[\beta^1(u), \ldots, \beta^n(u)]$

In Summary:

Given: a local frame $r_i = \sum_{i=1}^{n} R_i^j(u) \frac{\partial}{\partial u^j}$, $i = 1, \ldots, n$ on Ω.

Let: $R(u) := [R_1(u) \mid \cdots \mid R_n(u)]$, $\quad L(u) := R(u)^{-1}$

Jacobian problem Find all possible $\Lambda = \mathrm{diag}[\lambda^1, \ldots, \lambda^n]$, s.t.

$\boxed{J(u) = L^{-1}(u)\,\Lambda(u)\,L(u)}$ is a Jacobian matrix.

λ^i is an eignvalue of J with eigenvector-field r_i.

$f = (f^1, \ldots, f^n)$ such as $J = D_u f$ determined from J up to addition of a constant vector valued function

Hessian problem Find all possible $\mathcal{B} = \mathrm{diag}[\beta^1, \ldots, \beta^n]$ s.t.

$\boxed{H(u) = L^T(u)\,\mathcal{B}(u)\,L(u)}$ is a Jacobian matrix

β^i is the “length” of r_i relative to the inner product H.

A symmetric Jacobian H is a Hessian for some function η, determined from H up to an affine function

Scaling invariance

What happens if we replace $\mathcal{R} = (r_1, \ldots, r_n)$ with $\tilde{\mathcal{R}} = (\alpha_1 r_1, \ldots, \alpha_n r_n)$, where $\alpha_i \colon \Omega \to \mathbb{R}$ are non zero?

$R(u) := [R_1(u) \,|\, \cdots \,|\, R_n(u)]$ is component matrix of $\mathcal{R}$,
$L(u) := R(u)^{-1}$

$\tilde{R}(u) = R\mathcal{A}$ is component matrix of $\tilde{\mathcal{R}}$,
$\tilde{L}(u) := \tilde{R}(u)^{-1} = \mathcal{A}^{-1} L(u)$, where $\mathcal{A} = \mathrm{diag}[\alpha_i, \ldots, \alpha_n]$.

Jacobian problem: For all $\Lambda = \mathrm{diag}[\lambda^1, \ldots, \lambda^n]$,

$$J(u) = R(u)\,\Lambda(u)\,L(u) = \tilde{R}(u)\,\Lambda(u)\,\tilde{L}(u).$$

Λ solves the Jacobian problem for both $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

Hessian problem: For all $\mathcal{B} = \mathrm{diag}[\beta^1, \ldots, \beta^n]$,

$$H(u) = L^T(u)\,\mathcal{B}(u)\,L(u) = \tilde{L}^T(u)\,\tilde{\mathcal{B}}(u)\,\tilde{L}(u),$$

$\tilde{\mathcal{B}} := \mathrm{diag}[\alpha_1^2\,\beta^1, \ldots, \alpha_n^2 \beta^n]$ solves the Hessian problem $\Leftrightarrow$ $\mathcal{B}$ solves it.

$\mathcal{R}$ can be prescribed up to a scaling (integral curves of $\mathcal{R}$)

How many solutions?

A request for matrix J (or H) to be a Jacobian leads to an (overdetermined for $n > 2$) system of equations on λ's (or β's).

How many free constants and functions determine a general solution $\lambda^1(u), \dots, \lambda^n(u)$?

How many free constants and functions determine a general solution $\beta^1(u), \dots, \beta^n(u)$?

Goal: classify all possible scenarios depending on the properties of the frame $\mathcal{R}$.

Results:

- $n = 1, 2$ known
- $n = 3$ complete classification (Jenssen and K)
- $n > 3$ few known and few new results.

Conservative systems

$$u_t + f(u)_x = 0\,. \tag{1}$$

- one space-variable: $x \in \mathbb{R}$; one time-variable: $t \in \mathbb{R}$.
- $u(x,t) \in \Omega \subset \mathbb{R}^n$ (n equations on n unknown state functions).
- nonlinear *flux* $f(u)\colon \Omega \to \mathbb{R}^n$.

$$LHS(1) = u_t + (D_u f)\, u_x$$

(1) is *hyperbolic* if $Df(u)$ is diagonalizable over reals $\forall u \in \Omega$.
(1) is *strictly hyperbolic* if $\forall u \in \Omega$ eigenvalues of $Df(u)$ are real and distinct.

Example: the Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

  $v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, pressure $p(v, S) > 0$ is a given function of v and S, s.t $p_v < 0$.

- $U_t + f(U)_x = 0$,where $U = (v, u, S)$ and $f(U) = (-u, p(v, S), 0)$.

Wave curves

Self-similar solutions of $\boxed{u_t + f(u)_x = 0 \quad (*)}$

Smooth: (rarefaction curves)

$$u(x,t) \quad = \quad w\left(\frac{x}{t}\right) = w(\xi), \text{ where } \xi = \frac{x}{t}$$

$$\Downarrow (*)$$

$$[D_u f(w(\xi))]\ \dot{w}(\xi) \quad = \quad \xi \dot{w}(\xi), \text{ where } \dot{} = \frac{d}{d\xi}$$

$$\Downarrow$$

$\dot{w}(\xi)$ is an eigenvector of $D_u f$ with the eigenvalue ξ.

$$\Downarrow$$

through $\bar{u} \in \Omega$, $\exists$ n-solutions $w_i(\xi)$ which are eigencurves of r_i and $\xi = \lambda^i(w_i)$.

Discontinuous: (shock curves) are defined by Hugoniot locus $\{\, u \in \Omega \mid \exists s \in \mathbb{R} :\ f(u) - f(\bar{u}) = s \cdot (u - \bar{u}) \,\}$.

Cauchy problem:

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x).$$

In general, a solution will develop discontinuity even for smooth initial data — weak solutions.
Non uniqueness — admissibility criterion based on entropy inequality.

Riemann problem:

$$u_0(x) = \begin{cases} u_-\,, & x < 0 \\ u_+\,, & x > 0\,. \end{cases}$$

Lax (1957) under certain condition on f and when u_- and u_+ are close, solutions to Riemann problems are determined by wave curves.
Glimm (1965) for u_0 with small total variation, solutions to Cauchy problems is determined by solutions of Riemann problems.
Large initial data ???

Extensions and entropies: Assume that $\exists$ functions $q\colon \Omega \to \mathbb{R}$ and $\eta\colon \Omega \to \mathbb{R}$, s.t. $\boxed{\operatorname{grad} q = \operatorname{grad} \eta (D_u f)}$, then multiplication of $\boxed{u_t + f(u)_x = 0}$ by $(\operatorname{grad} \eta)$ from the left (assuming that u is smooth) leads to a companion conservation law:

$$\eta(u)_t + q(u)_x = 0$$

η is called an extension of conservative system.

Proposition: η is an extension iff:

for each pair $1 \leq i \neq j \leq n$: $\boxed{\lambda^j = \lambda^i \quad \text{or} \quad R_i^T \left(D_u^2 \eta\right) R_j = 0}$.

An extension η is called an entropy if $D_u^2 \eta$ is positive semidefinite and is called strict entropy if $D_u^2 \eta$ is positive definite.

Admissibility criterion:

A weak solution of $u_t + f(u)_x = 0$ is admissible if it is a limit of smooth solutions

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \,, \qquad \text{as } \varepsilon \downarrow 0.$$

If η is an entropy with flux q, then:

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leq \varepsilon \eta(u^\varepsilon)_{xx} \qquad (\varepsilon > 0)$$

A weak solution of $u_t + f(u)_x = 0$ is *admissible* if it satisfies the entropy inequality

$$\eta(u)_t + q(u)_x \leq 0 \qquad \text{(distributional sense)}$$

Solution of the Jacobian problem:

$$R(u)\,\Lambda(u)\,L(u) \text{ is a Jacobian}$$

... and the λ-system.

Trivial solutions of the Jacobian problem

- $\forall \mathcal{R}$: $\exists$ one-parameter family of trivial solutions $\lambda^1(u) = \cdots = \lambda^n(u) \equiv \bar{\lambda}$, where $\bar{\lambda} \in \mathbb{R}$:

$$R(u)\,\bar{\Lambda}\,L(u) = \bar{\Lambda} = Df \text{ for } f = \bar{\lambda}u + \bar{u},\ \bar{u} \in \mathbb{R}^n.$$

- $\exists \mathcal{R}$ with only trivial solutions. Example:

$$R_1 = [u^1,\, u^2,\, 0]^T,\ R_2 = [-u^2,\, u^1,\, 0]^T,\ R_3 = [-u^2,\, u^1,\, 1]^T$$

- $\lambda^1(u) = \cdots = \lambda^n(u)$ is a solution

$$\Updownarrow$$

$$\lambda^1(u) = \cdots = \lambda^n(u) \equiv \bar{\lambda} \text{ for some } \bar{\lambda} \in \mathbb{R}.$$

Direct Formulation

- A matrix $J(u) = (J^i_j(u))$ is a Jacobian

$$\frac{\partial J^i_j(u)}{\partial u^k} = \frac{\partial J^i_k(u)}{\partial u^j} \text{ for all } i, j, k = 1, \dots, n \text{ with } j < k\,,$$

- $J(u) = R(u)\Lambda(u)L(u)$ is a Jacobian

$$\Updownarrow$$

$$\sum_{m=1}^{n} \Big[C^i_{mj}\partial_k\lambda^m - C^i_{mk}\partial_j\lambda^m + \lambda^m\big(\partial_k C^i_{mj} - \partial_j C^i_{mk}\big)\Big] = 0\,,$$

$$i, j, k = 1, \dots, n \text{ with } j < k,$$

where $C^i_{mj}(u) := R^i_m(u)L^m_j(u), \qquad \partial_i = \frac{\partial}{\partial u_i}$

- A linear, variable coefficient system of $\frac{n^2(n-1)}{2}$ first order PDEs for n unknowns $\lambda^1, \dots, \lambda^n$.

Formulation in terms of differential forms

$$J(u) \text{ is a Jacobian matrix} \qquad \Longleftrightarrow \qquad dJ(u) \wedge du = 0\,,$$

where $du := (du^1, \ldots, du^n)^T$.

$J(u) = R(u)\,\Lambda(u)\,L(u)$ is a Jacobian

$$\Updownarrow$$

$$\{L(dR)\Lambda + d\Lambda - \Lambda L(dR)\} \wedge L du = 0\,.$$

(LHS is an n-vector of differential two-forms)

Rewriting in terms of the given frame:

- $r_i(u) := \sum_{m=1}^n R_i^m(u)\frac{\partial}{\partial u^m}$ is given frame
- $\ell^i(u) := \sum_{m=1}^n L_m^i(u)du^m$ is the dual coframe.
- $\ell := (\ell^1, \dots, \ell^n)^T$
- $$[r_i, r_j] = \sum_{k=1}^n c_{ij}^k \, r_k, \quad d\ell^k = -\sum_{i<j} c_{ij}^k \, \ell^i \wedge \ell^j \,.$$
- $\Gamma_{ij}^k := L^k(DR_j)R_i$ is the Christoffel symbols of the connection $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$ computed relative to the frame $\{r_1, \dots, r_n\}$ i.e. $\nabla_{r_i} r_j = \sum_{k=1}^n \Gamma_{ij}^k r_k.$
- Matrix $\mu := LdR$ of connection forms: $\mu_j^k = \sum_{i=1}^n \Gamma_{ij}^k \ell^i \,.$

$$\big(L(dR)\Lambda + d\Lambda - \Lambda L(dR)\big) \wedge Ldu = 0$$

$$\Updownarrow$$

$$\big(\mu\Lambda + d\Lambda - \Lambda\mu\big) \wedge \ell = 0 \,.$$

$$\Updownarrow$$

Differential/Algebraic system (the λ-system)

$$\left[\left(\mu\Lambda + d\Lambda - \Lambda\mu\right)\wedge \ell\right](r_i, r_j) \quad = \quad 0 \text{ for } 1 \le i < j \le n$$
$$\Updownarrow$$

$\lambda(\mathcal{R})$-system: $n(n-1)$ linear, homogeneous, 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ algebraic equations.

$$\begin{cases} r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j) & i \ne j, \qquad (\lambda(\mathcal{R})\text{-diff}) \\ \Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k) & i < j,\, i \ne k,\, j \ne k \qquad (\lambda(\mathcal{R})\text{-alg}) \end{cases}$$

$n = 1$ – $\lambda(\mathcal{R})$ is empty
$n = 2$ – $\lambda(\mathcal{R})$-alg is empty

In different contexts the λ-system appeared in Sévannec (1994), Tsarëv (1985)

Symmetry and flatness

$$d\ell = -\mu\wedge\ell \qquad \text{(Symmetry)}, \qquad d\mu = -\mu\wedge\mu \qquad \text{(Flatness)}.$$

$$\Updownarrow$$

$$c^i_{km} = \Gamma^i_{km} - \Gamma^i_{mk} \qquad \text{(Symmetry)}$$

and

$$r_m\left(\Gamma^j_{ki}\right) - r_k\left(\Gamma^j_{mi}\right) = \sum_{s=1}^{n}\left(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\right) \;\; \text{(Flatness)}.$$

The rank of the algebraic part:

$\Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k)$, $i < j$, $i \neq k$, $j \neq k$. $(\lambda(\mathcal{R})\text{-alg})$

Observation: $0 \leq \text{rank}(\lambda(\mathcal{R})\text{-alg}) \leq (n-1)$.

Extreme cases:

$$\text{rank}(\lambda(\mathcal{R})\text{-alg}) = (n-1) \Rightarrow \lambda^1(u) = \cdots = \lambda^n(u) \equiv \bar{\lambda} \in \mathbb{R}$$

only trivial solutions

$$\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0 \Leftrightarrow \Gamma^k_{ji} = 0,\ \forall i, j, k \text{ distinct } \Rightarrow c^k_{ji} = 0$$

$\forall i, j, k$ distinct $\Leftrightarrow$ $\boxed{[r_i, r_j] \in \text{span}\{r_i, r_j\} \text{ (rich frame)}}$.

- only trivial solutions $\not\Rightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = n - 1$.
- $\mathcal{R}$ is rich $\not\Rightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0$.
- we will show:
  $\mathcal{R}$ is rich and admits strictly hyperbolic conservative systems
  $\Rightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0$.

Rich frame

- **Definition** A frame $r_1, \ldots, r_n$ is *rich* if each pair of vector-fields is in involution, i. e. $\forall 1 \leq i, j \leq n$:

$$[r_i, r_j] = c^i_{ij} r_i + c^j_{ij} r_j \quad \Leftrightarrow \quad c^k_{ij} = 0 \quad k \neq i,\ k \neq j.$$

$$\Downarrow$$

- $\exists$ smooth functions $\alpha^i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ such that $\tilde{r}_1 := \alpha^1 r_1, \ldots, \tilde{r}_n := \alpha^n r_n$ commute.

$$\Downarrow$$

- $\exists$ a change of coordinates

$$(w^1(u), \ldots, w^n(u)) = \rho(u)$$

s.t. $\quad \tilde{r}_i = \frac{\partial}{\partial w^i}, \quad i = 1, \ldots, n.$

$(w^1(u), \ldots, w^n(u))$ are called Riemann coordinates.

λ-system in Riemann coordinates $(w^1(u), \dots, w^n(u)) = \rho(u)$

$$\partial_i \lambda^j(w) = \Gamma^j_{ji}(w)\,(\lambda^i(w) - \lambda^j(w)) \quad \text{for} \quad i \neq j,$$

$$\text{where } \partial_i = \frac{\partial}{\partial w^i}$$

$$\Gamma^k_{ij}(w)\,(\lambda^j(w) - \lambda^i(w)) = 0 \qquad \text{for} \quad i < j,\ k \neq i,\ k \neq j$$

- $\forall$ distinct i, j, k: $\Gamma^k_{ij} = 0 \quad \Rightarrow \quad$ algebraic part is empty
- $\exists$ distinct i, j, k s.t. $\Gamma^k_{ij} \neq 0 \quad \Rightarrow \quad$ multiplicity conditions on eigenvalues are implied by the system.

Flatness and symmetries in Riemann coordinates

$$\Gamma^i_{km}(w) = \Gamma^i_{mk}(w) \quad \text{(Symmetry)}$$

$$\partial_m\big(\Gamma^j_{ki}\big) - \partial_k\big(\Gamma^j_{mi}\big) = \sum_{s=1}^{n}\big(\Gamma^j_{ks}\,\Gamma^s_{mi} - \Gamma^j_{ms}\,\Gamma^s_{ki}\big) \qquad \text{(Flatness)}.$$

Rich frame with empty $\lambda(\mathcal{R})$-alg

$$\partial_i \lambda^j = \Gamma^j_{ji}(\lambda^i - \lambda^j) \text{ for } 1 \leq i \neq j \leq n, \qquad \partial_i := \frac{\partial}{\partial w_i}.$$

- Compatibility conditions $\partial_k \partial_m \lambda^j = \partial_m \partial_k \lambda^j$, where the first derivatives $\partial_i \lambda^j$, $i = 1, \ldots, n$ are given by the equations, are met due to the flatness of the connection.
- Darboux theorem $\Rightarrow$ general solution depends on n functions of one variable $\phi^i(w^i)$, $i = 1, \ldots, n$ s.t. for $\bar{w} \in \Omega$

$$\lambda^i(\bar{w}^1, \ldots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1}, \ldots, \bar{w}^n) = \phi^i(w^i).$$

- all $n = 2$ frames belong to this case.
- rich orthogonal frames belong to this case.

Example: rich orthogonal frame (cylindrical coordinates)

$$R_1 = [u^1,\, u^2,\, 0]^T, \quad R_2 = [-u^2,\, u^1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

Riemann coordinates: (in the first octant):

$$w^1 = \tfrac{1}{2}\ln\left[(u^1)^2 + (u^2)^2\right], \quad w^2 = \arctan\left(\frac{u^2}{u^1}\right), \quad w^3 = u^3\,.$$

$$\lambda^1 = \psi_1(w^1), \quad \lambda^2 = e^{-w^1}\int_*^{e^{w^1}} \psi_1(\ln(\tau^2))\,d\tau + e^{-w^1}\,\psi_2(w^2),$$
$$\lambda^3 = \psi_3(w^3).$$

Rich system with non-trivial algebraic constraints

$$\partial_i \lambda^j = \Gamma^j_{ji}(\lambda^i - \lambda^j) \quad \text{for} \quad 1 \le i \ne j \le n, \quad \partial_i := \frac{\partial}{\partial w_i}.$$
$$\Gamma^k_{ij}(\lambda^j - \lambda^i) = 0 \quad \text{for} \quad 1 \le k \ne i < j \ne k \le n.$$

- $\exists$ distinct i, j, k s.t. $\Gamma^k_{ij} \ne 0$
- multiplicity conditions on eigenvalues are implied by the algebro-differential system (no strictly hyperbolic conservation laws in this case).
- Darboux theorem $\Rightarrow$ general solution depends on s_0 constants and s_1 functions of one variable, where
  - s_0 is the number of distinct eigenvalues of multiplicity > 1,
  - s_1 is the number of eigenvalues of multiplicity 1.

$\lambda(\mathcal{R})$-system for $n = 3$

I. rank($\lambda(\mathcal{R})$-alg) $= 0 \Rightarrow \mathcal{R}$ is rich; a general solution of $\lambda(\mathfrak{R})$ depends on 3 functions of 1 variable; $\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$.

II. rank($\lambda(\mathcal{R})$-alg) $= 1$ (a single algebraic constraint):

IIa. All three λ^i appear in the algebraic constraint $\Rightarrow \lambda(\mathcal{R})$ can be analyzed by Fronebious theorem; the solution of the λ-system is either trivial or depends on 2 arbitrary constants; In the latter case, $\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$; $\nexists$ rich systems in class IIa.

IIb. Exactly two λ^i appear in the algebraic constraint $\Rightarrow$ two λ^i coincide; $\lambda(\mathcal{R})$ can be analyzed by Cartan-Kähler theorem; the general solution is either trivial or depends on 1 arbitrary function of 1 variables and 1 constant; $\nexists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$; but $\exists$ rich systems, in class IIb.

III. rank($\lambda(\mathcal{R})$-alg) $= 2 \Rightarrow$ only trivial solutions $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \bar{\lambda} \in \mathbb{R}$.

Solution of the Hessian problem

$$L^T(u)\,\mathcal{B}(u)\,L(u) \text{ is a Jacobian}$$

... and the β-system.

Trivial solutions of the Hessian problem

- $\forall\mathcal{R}$: $\exists$ a trivial solutions $\beta^1(u) = \cdots = \beta^n(u) \equiv 0$

$$(0) = D_u^2\eta \text{ for } \eta(u) = a \cdot u + b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}.$$

- $\exists\mathcal{R}$ with only trivial solutions. Example:

$$R_1 = [u^1,\, -u^2,\, 0]^T,\ R_2 = [-u^1,\, u^2,\, 1]^T,\ R_3 = [1,\, 1,\, 1]^T.$$

- $\lambda(\mathcal{R})$ has non trivial solutions (class IIa)

$$\lambda^1 = \lambda^2 = C,\ \lambda^3 = (u^1 + u^2)\, F(u^1\, u^2)$$

The β-system

$$L^T(u)\,\mathcal{B}(u)\,L(u) \qquad \text{is a Jacobian}$$

$$\Updownarrow$$

$\beta(\mathcal{R})$-system: $n(n-1)$ linear, homogeneous, 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ algebraic equations.

$$\begin{cases} r_i(\beta^j) = \beta^j\,(\Gamma^j_{ij} + c^j_{ij}) - \beta^i\,\Gamma^i_{jj} & i \neq j, & \beta(\mathcal{R})\text{-diff} \\ \beta^k\,c^k_{ij} + \beta^j\Gamma^j_{ik} - \beta^i\,\Gamma^i_{jk} = 0 & i < j,\, i \neq k,\, j \neq k & \beta(\mathcal{R})\text{-alg} \end{cases}$$

$n = 1$ – $\beta(\mathcal{R})$ is empty
$n = 2$ – $\beta(\mathcal{R})$-alg is empty

In a different context the β-system appeared in Conlon and Liu (1981)

The rank of algebraic part:

$$\beta^k c^k_{ij} + \beta^j \Gamma^j_{ik} - \beta^i \Gamma^i_{jk} = 0, \; i < j, \; i \neq k, \; j \neq k. \quad (\beta(\mathcal{R})\text{-alg})$$

- $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 0 \Leftrightarrow \Gamma^k_{ji} = 0, \forall i, j, k$ distinct $\Leftrightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0 \Rightarrow c^k_{ji} = 0 \; \forall i, j, k$ distinct $\Leftrightarrow$ $\boxed{[r_i, r_j] \in \text{span}\{r_i, r_j\} \text{ (rich frame)}}$.
- $\text{rank}(\beta(\mathcal{R})\text{-alg}) = \text{rank}(\lambda(\mathcal{R})\text{-alg})$ for $n \leq 3$.
- in general $\text{rank}(\beta(\mathcal{R})\text{-alg}) \neq \text{rank}(\lambda(\mathcal{R})\text{-alg})$ for $n > 3$ ($\exists \, (n = 4)$- example).

Rich frame: β-system in Riemann coordinates

$$(w^1(u), \ldots, w^n(u)) = \rho(u)$$

$$\partial_i \beta^j = \Gamma^j_{ji}\,\beta^j - \Gamma^i_{jj}\,\beta^i \qquad \text{for } i \neq j, \qquad \left(\partial_i = \frac{\partial}{\partial w^i}\right)$$

$$\Gamma^j_{ik}\beta^j = \Gamma^i_{jk}\beta^i \qquad \text{for } i < j,\ k \neq i,\ k \neq j,$$

Case: rank($\beta(\mathcal{R})$-**alg**) $= 0$: $\forall$ distinct i, j, k: $\Gamma^k_{ij} = 0 \Leftrightarrow$ no algebraic constraints $\Rightarrow$ a differential system of Darboux type $\Rightarrow$ general solution depends on n functions of one variable $\phi^i(w^i)$, $i = 1, \ldots, n$ s.t. for $\bar{w} \in \Omega$

$$\beta^i(\bar{w}^1, \ldots, \bar{w}^{i-1}, w^i, \bar{w}^{i+1}, \ldots, \bar{w}^n) = \phi^i(w^i).$$

$\beta(\mathcal{R})$-system for $n = 3$

$$0 \leq \text{rank}(\beta(\mathcal{R})\text{-alg}) = \text{rank}(\lambda(\mathcal{R})\text{-alg}) \leq 2.$$

I. rank($\beta(\mathcal{R})$-alg) $= 0 \Rightarrow \mathcal{R}$ is rich; a general solution of $\beta(\mathfrak{R})$ depends on 3 functions of 1 variable; $\exists$ hyperbolic conservative system with eigenframe $\mathcal{R}$, each of them posses strict entropies.

II. rank($\beta(\mathcal{R})$-alg) $= 1$ (a single algebraic constraint): classification on the next page

III. rank($\beta(\mathcal{R})$-alg) $= 2 \Rightarrow$ only trivial solutions of $\lambda(\mathcal{R})$: $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \bar{\lambda} \in \mathbb{R}$ with the flux $f = \bar{\lambda} u + \bar{u}$, $\bar{u} \in \mathbb{R}^n$. The size of the solution of $\beta(\mathcal{R})$-system may vary, but any function η is an extension because the first condition is satisfied:

$$\forall i \neq j: \quad \boxed{\lambda^j = \lambda^i \quad \text{or} \quad R_i^T \left(D_u^2 \eta\right) R_j = 0}.$$

Solutions for $\beta(\mathcal{R})$-system when $n = 3$ and rank($\beta(\mathcal{R})$-alg) $= 1$.

(1) Only the trivial solution: $\beta^1 = \beta^2 = \beta^3 \equiv 0$ *($\mathcal{R}$ may be rich)*

(2) Exactly two β^i are zero and the third depends on 1 arbitrary function of one variable. *($\mathcal{R}$ may be rich)*

(3) Exactly one β^i is zero and the other two β^i depend on

- **(3a)** 2 arbitrary functions of one variable. *($\mathcal{R}$ may be rich)*
- **(3b)** 1 common arbitrary constant.

(4) There are non-degenrate solutions (all β^i are non-zero) which depends on

- **(4a)** 1 arbitrary function of one variable and 1 arbitrary constant.
- **(4b)** 2 arbitrary constants.
- **(4c)** 1 arbitrary constant.

The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

$v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $\boxed{p(v,S) > 0}$ is pressure as a given function of v and S, s.t $\boxed{p_v < 0}$.

- $U_t + f(U)_x = 0$,where $U = (v, u, S)$ and $f(U) = (-u, p(v,S), 0)$.
- eigenvalues of Df are $\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}$.
- eigenvectors of D_f are $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $R_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

Inverse problem:

- For a given pressure function $p = p(v, S) > 0$, with $p_v < 0$ define a frame $\mathcal{R}$: $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $R_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$
- determine the class of conservative systems with eigenfields $\mathcal{R}$ by solving the λ-system for λ^1, λ^2, λ^3.
- Observation: frame is rich $\Leftrightarrow$ $\left(\frac{p_S}{p_v}\right)_v \equiv 0 \Leftrightarrow p(v, S) = \Pi(v + F(S))$.

Solution of the $\lambda(\mathcal{R})$-system:

in the non-rich case:

- $\lambda(\mathcal{R})$-alg consists of:

$$\frac{p_v}{4}\left(\frac{p_S}{p_v}\right)_v(\lambda^1+\lambda^3-2\lambda^2)=0 \Leftrightarrow \lambda^2=\frac{1}{2}(\lambda^1+\lambda^3)$$

  that involves all three λ's (case IIa) $\Rightarrow$ the general solution depends on two constants.

- from the differential part of λ-system we obtain:

$$\lambda^1 = C_1 - C_2\sqrt{-p_v}\,, \quad \lambda^2 \equiv C_1\,, \quad \lambda^3 = C_1 + C_2\sqrt{-p_v}\,.$$

in the rich case:

- $\lambda(\mathcal{R})$ is empty $\Rightarrow$ solution depends on 3 arbitrary functions of one variable.

Solution of the $\beta(\mathcal{R})$-system:

in the non-rich case:

- $\beta(\mathcal{R})$-alg consists of:

$$\left(\frac{p_S}{p_v}\right)_v(\beta^1-\beta^3)=0 \Leftrightarrow \beta^1=\beta^3$$

- The general solution depends on 1 function of 1 variable and 1 constant (case 4a):

$$\begin{aligned}\beta^1 &= \beta^3 = K_1\, p_v\,,\\ \beta^2 &= \frac{K_1 p_v^2}{2}\left(\int_{K_2}^{v} p_{SS}(\tau,S)\,d\tau-\frac{p_S^2}{p_v}(v,S)+F(S)\right).\end{aligned}$$

  (K_2 can be absorbed into arbitrary function)
- $\exists$ strict entropies.

in the rich case:

- $\beta(\mathcal{R})$-alg is empty $\Rightarrow$ solution depends on 3 arbitrary functions in one variable.

Blowup example

$$R_1 = [-1,\, 0,\, u^2+1]^T\,, \qquad R_2 = \left[\frac{u^3}{(u^2)^2-1},\, -1,\, u^1\right]^T,$$

$$R_3 = [1,\, 0,\, 1-u^2]^T\,.$$

- non-rich frame
- $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = \text{rank}(\beta(\mathcal{R})\text{-alg}) = 1$

Solution of the λ-system

$\lambda(\mathcal{R})$-alg: $2\lambda^2 = (1-u^2)\lambda^1 + (1+u^2)\lambda^3$ involves all three λ's (case IIa) $\Rightarrow$ the general solution depends on two constants:

$$\lambda^1 = C_1 - 2\,C_2, \qquad \lambda^2 = C_1 + (u^2-1)\,C_2, \qquad \lambda^3 = C_1\,.$$

fluxes:

$$f(u) = \begin{pmatrix} (C_1 + C_2\,(u^2-1))\,u^1 + C_2\,u^3, \\ u^2\,(C_1 - C_2 + \frac{1}{2}C_2\,u^2), \\ C_2\,u^1\,(1-(u^2)^2) - C_2\,u^2\,u^3 + (C_1 - C_2)\,u^3 \end{pmatrix}.$$

Solution of the β-system

$\beta(\mathcal{R})$-alg: $(u^2 - 1)\beta^1 = (u^2 + 1)\beta^3$ the general solution depends on one arbitrary function of one variable:

$$\beta^1 \equiv 0, \qquad \beta^2 = F(u^2), \qquad \beta^3 \equiv 0,$$

extensions (modulo affine parts):

$$\eta(u^1, u^2, u^3) = G(u^2)\,, \qquad \text{where } G'' = F.$$

Rich orthogonal frame

$$R_1 = [u^1,\, u^2,\, 0]^T, \quad R_2 = [-u^2,\, u^1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

$\lambda(\mathcal{R})$-alg and $\beta(\mathcal{R})$-alg are empty $\Rightarrow$ general solutions of $\lambda(\mathcal{R})$ and $\beta(\mathcal{R})$ depend on 3 arbitrary functions of 1 variable:

$$\lambda^1 = F_1(v), \qquad \lambda^2 = \frac{1}{\sqrt{v}} \int_*^{\sqrt{v}} F_1(\tau^2)\, d\tau + \frac{1}{u^1}\, F_2\left(\frac{u^2}{u^1}\right),$$

$$\lambda^3 = F_3(u^3);$$

$$\beta^1 = v\, G_1(v), \qquad \beta^2 = \sqrt{v} \int_*^{\sqrt{v}} G_1(\tau^2)\, d\tau + u^1\, G_2\left(\frac{u^2}{u^1}\right),$$

$$\beta^3 = G_3(u^3), \qquad \text{where } v = v = (u^1)^2 + (u^2)^2.$$

---

any solution of λ-system can be combined with any solution of β-system.

Jacobian structures

- M is a manifold with a flat connection ∇.
- $J\colon \mathcal{X}(M) \to \mathcal{X}(M)$ is called a *Jacobian* if $\exists$ a vector field $V \in \mathcal{X}(M)$ s. t.

$$J(X) = \nabla_X V, \quad \forall X \in \mathcal{X}(M).$$

  We then use notation J_V.
- J is a Jacobian $\Rightarrow$

$$\boxed{\nabla_X J(Y) - \nabla_Y J(X) = J([X,Y]) \quad \forall X, Y \in \mathcal{X}(M) \ (*)}$$

- if $(u^1, \dots, u^n)$ are affine coordinates and $V = \sum_{i=1}^{n} f^i(u) \frac{\partial}{\partial u^i}$, then $J_V(\frac{\partial}{\partial u^j}) = \sum_i^n \frac{\partial f^i}{\partial u^j} \frac{\partial}{\partial u^i}$
- if $\mathcal{R} = (r_1, \dots, r_n)$ is an eigenframe of a Jacobian map: $\boxed{J(r_i) = \lambda^i r_i}$ for $\lambda^i\colon M \to \mathbb{R}$, then (*) evaluated on $X = r_i, Y = r_j \Rightarrow$ the λ-system.

Hessian structures

- M is a manifold with a flat connection ∇.
- a metric g on M is called a *Hessian* if $\exists$ a function $\nu\colon M \to \mathbb{R}$ s. t. $\Leftrightarrow \forall X, Y \in \mathcal{X}(M)$.:

$$g(X,Y) = (\nabla_X d\eta)(Y) := X(d\eta(Y)) - d\,\eta(\nabla_X Y)$$

  We then use notation g_η.

- g is a Hessian $\Rightarrow$

$$\boxed{(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z), \quad \forall X, Y, Z \in \mathcal{X}(M) \ (*)}$$

- if $(u^1, \dots, u^n)$ are affine coordinates then $g_\eta(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = \frac{\partial^2 \eta}{\partial u^i\, \partial u^j}$
- if $\mathcal{R} = (r_1, \dots, r_n)$ is orthogonal: $\boxed{g(r_i, r_j) = \delta^i_j \beta^i}$ for $\beta^i\colon M \to \mathbb{R}$, then $(*)$ evaluated on $X = r_i, Y = r_j, Z = r_k \Rightarrow$ the β-system.

Sévennec's problem: For a given quasilinear system

$$v_t + A(v)v_x = 0\,,$$

Sévennec shows that there is a coordinate system in which the system is conservative if and only if there exists a flat and symmetric affine connection ∇ such that the eigenvalues of A satisfy

$$\begin{aligned} r_i(\lambda^j) &= \Gamma^j_{ji}(\lambda^i - \lambda^j) \qquad \text{for } i \neq j, \\ (\lambda^i - \lambda^k)\Gamma^k_{ji} &= (\lambda^j - \lambda^k)\Gamma^k_{ij} \qquad \text{for } i < j,\, i \neq k,\, j \neq k., \end{aligned}$$

where Γ^j_{ji} are the Christoffel symbols of ∇ relative to the eigenframe of A.