CARTAN'S GENERALIZATION OF LIE'S THIRD THEOREM

ROBERT L. BRYANT

MATHEMATICAL SCIENCES RESEARCH INSTITUTE

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In many ways, this talk (and much of the work it reports on) owes its existence to an article

I. M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan. Part I (the transitive groups)*, Journal D'Analyse Math. **15** (1965), 1–114.

'Simple' examples (now called *pseudogroups*) include:

(1) Diffeomorphisms
(2) Volume-preserving diffeomorphisms
(3) Symplectomorphisms and Contactomorphisms
(4) Biholomorphisms, etc.

Part II never appeared, and so there was no mention of gauge groups (one of Cartan's examples of an *intransitive* infinite simple group), and some of the more delicate issues that Cartan faced (with only partial success) were avoided.

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Cartan's work on the intransitive case has been (partially) taken up by many different people, and it has given rise to the theory of what are now called *Lie algebroids* and *Lie groupoids*.

Lie's Third Theorem: If L is a finite-dimensional, real Lie algebra, then there exists a Lie algebra homomorphism $\lambda : L \to \mathrm{Vect}(L)$ satisfying

$$\lambda(x)(0) = x \qquad \text{for all } x \in L.$$

Dual Formulation: Let $\delta : L^* \to \Lambda^2(L^*)$ be a linear map. If its extension $\delta : \Lambda^*(L^*) \to \Lambda^*(L^*)$ as a graded derivation of degree 1 satisfies $\delta^2 = 0$, then there is a DGA homomorphism $\phi : \big(\Lambda^*(L^*), \delta\big) \to \big(\Omega^*(L), \mathrm{d}\big)$ satisfying

$$\phi(\alpha)(0) = \alpha.$$

Basis Formulation: If $C^i_{jk} = -C^i_{kj}$ $(1 \le i, j, k \le n)$ are constants, then there exist linearly independent 1-forms ω^i $(1 \le i \le n)$ on $\mathbb{R}^n$ satisfying the structure equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k$$

if and only if these formulae imply $\mathrm{d}\big(\mathrm{d}\omega^i\big) = 0$ as a formal consequence.

A geometric problem: Classify those Riemannian surfaces (M^2, g) whose Gauss curvature K satisfies the second order system

$$\mathrm{Hess}_g(K) = a(K)g + b(K)\mathrm{d}K^2$$

for some functions a and b.

Analysis Writing $g = {\omega_1}^2 + {\omega_2}^2$, the structure equations yield

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2 & \qquad \mathrm{d}\omega_{12} &= K\,\omega_1 \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \quad \omega_{12} \wedge \omega_1 & \mathrm{d}K &= K_1\,\omega_1 + K_2\,\omega_2 \end{aligned}$$

and the condition to be studied is encoded as

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\,{K_1}^2 & b(K)\,K_1 K_2 \\ b(K)\,K_1 K_2 & a(K) + b(K)\,{K_1}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Applying $\mathrm{d}^2 = 0$ to these two equations yields

$$\bigl(a'(K) - a(K)b(K) + K\bigr)\,K_i = 0 \qquad \text{for } i = 1, 2.$$

Thus, unless $a'(K) = a(K)b(K) - K$, such metrics have K constant.

Conversely, suppose that $a'(K) = a(K)b(K) - K$. Does there exist a 'solution' (N^3, ω) to the following system?

$$
\begin{aligned}
d\omega_1 &= -\omega_{12} \wedge \omega_2 \\
d\omega_2 &= \omega_{12} \wedge \omega_1 \\
d\omega_{12} &= K\, \omega_1 \wedge \omega_2
\end{aligned}
\qquad \omega_1 \wedge \omega_2 \wedge \omega_{12} \neq 0, \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_{12} \end{pmatrix}
$$

$$
\begin{pmatrix} dK \\ dK_1 \\ dK_2 \end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 0 \\ a(K) + b(K)\, {K_1}^2 & b(K)\, K_1 K_2 & -K_2 \\ b(K)\, K_1 K_2 & a(K) + b(K)\, {K_1}^2 & K_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_{12} \end{pmatrix}.
$$

Note: $d^2 = 0$ is 'formally satisfied' for these structure equations.

Answer: A theorem of É. Cartan [1904] implies that a 'solution' (N^3, ω) does indeed exist and is determined uniquely (locally near p, up to diffeomorphism) by the 'value' of (K, K_1, K_2) at p.

Cartan's result: Suppose that $C^i_{jk} = -C^i_{kj}$ and F^α_i (with $1 \le i, j, k \le n$ and $1 \le \alpha \le s$) are real-analytic functions on $\mathbb{R}^s$ such that the equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\, \omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a)\, \omega^i$$

formally satisfy $\mathrm{d}^2 = 0$. Then, for every $b_0 \in \mathbb{R}^s$, there exists an open neighborhood U of $0 \in \mathbb{R}^n$, linearly independent 1-forms η^i on U, and a function $b : U \to \mathbb{R}^s$ satisfying

$$\mathrm{d}\eta^i = -\tfrac{1}{2} C^i_{jk}(b)\, \eta^j \wedge \eta^k, \qquad \mathrm{d}b^\alpha = F^\alpha_i(b)\, \eta^i, \qquad \text{and} \qquad b(0) = b_0.$$

Up to local diffeomorphism of $(\mathbb{R}^n, 0)$, the pair (η, b) is germ-unique.

Remark 1: Cartan assumed that $F = (F^\alpha_i)$ has constant rank, but it turns out that, for a 'solution' (η, b) with U connected, $F(b) = \big(F^\alpha_i(b)\big)$ always has constant rank anyway.

Remark 2: Cartan worked in the real-analytic category and used the Cartan-Kähler theorem in his proof, but the above result is now known to be true in the smooth category. (Cf. P. Dazord)

The Hessian equation example:

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12}\wedge\omega_2 \\ \mathrm{d}\omega_2 &= \quad \omega_{12}\wedge\omega_1 \\ \mathrm{d}\omega_{12} &= K\,\omega_1\wedge\omega_2 \end{aligned} \qquad \omega_1\wedge\omega_2\wedge\omega_{12}\neq 0, \qquad \omega = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_{12}\end{pmatrix}$$

$$\begin{pmatrix}\mathrm{d}K\\ \mathrm{d}K_1\\ \mathrm{d}K_2\end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 0\\ a(K)+b(K)\,{K_1}^2 & b(K)\,K_1K_2 & -K_2\\ b(K)\,K_1K_2 & a(K)+b(K)\,{K_1}^2 & K_1 \end{pmatrix} \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_{12}\end{pmatrix}.$$

$\mathrm{d}^2=0$ is formally satisfied when $a'(K)=a(K)b(K)-K$.

Remark: The F-matrix either has rank 0 (when $K_1=K_2=a(K)=0$) or 2 (all other cases). The rank 0 cases have K constant. The rank 2 cases have a 1-dimensional symmetry group and each represents a surface of revolution.

Modern formulation of Cartan's theory: A *Lie algebroid* is a vector bundle $a : Y \to A$ over a manifold A endowed with a Lie algebra structure

$$\{,\} : \Gamma(Y) \times \Gamma(Y) \to \Gamma(Y)$$

and a bundle map $\alpha : Y \to TA$ that induces a Lie algebra homomorphism on sections and satisfies the Leibnitz compatibility condition

$$\{U, fV\} = \alpha(U)(f)\,V + f\,\{U, V\} \qquad \text{for } f \in C^\infty(A) \text{ and } U, V \in \Gamma(Y).$$

In our case, take a basis U_i of $Y = \mathbb{R}^s \times \mathbb{R}^n$ with $a : Y \to \mathbb{R}^s$ the projection and define

$$\{U_j, U_k\} = C^i_{jk}(a)\,U_i \qquad \text{and} \qquad \alpha(U_i) = F^\alpha_i(a)\,\frac{\partial}{a^\alpha}.$$

The 'formal' condition $\mathrm{d}^2 = 0$ ensures that α induces a Lie algebra homomorphism and satisfies the above compatibility condition.

A 'solution' is a $b : B^n \to A$ covered by a bundle map $\eta : TB \to Y$ of rank n that induces a Lie algebra homomorphism on sections.

Some geometric applications:

1. Classification of Bochner-Kähler metrics. (B—, 2001)

2. Classification of Levi-flat minimal hypersurfaces in $\mathbb{C}^2$. (B—, 2002)

3. Classification of isometrically deformable surfaces preserving the mean curvature. (originally done by Bonnet in the 1880s)

4. Surfaces with prescribed curvature operator. (B—, 2001)

5. Classification of projectively flat Finsler surfaces of constant flag curvature (B—, 1997).

6. Classification of austere 3-folds. (B—, 1991)

7. Classification of the exotic symplectic holonomies. (originally done by Schwachhofer, et al.)

Many more examples drawn from classical differential geometry.

A generalization of Cartan's Theorem. Consider equations

$$\mathrm{d}\eta^i = -\tfrac{1}{2}C^i_{jk}(h)\,\eta^j \wedge \eta^k \qquad \mathrm{d}h^a = \bigl(F^a_i(h) + A^a_{i\alpha}(h)p^\alpha\bigr)\,\eta^i.$$

C^i_{jk}, F^a_i, and $A^a_{i\alpha}$ (where $1 \le i, j, k \le n$, $1 \le a \le s$, and $1 \le \alpha \le r$) are specified functions on a domain $X \subset \mathbb{R}^s$. Assume:

(1) The functions C, F, and A are real analytic.
(2) The tableau $A(h) = \bigl(A^a_{i\alpha}(h)\bigr)$ is rank r and *involutive*, with Cartan characters $s_1 \ge s_2 \ge \cdots \ge s_q > s_{q+1} = 0$ for all $h \in \mathbb{R}^s$.
(3) $\mathrm{d}^2 = 0$ reduces to equations of the form

$$0 = A^a_{i\alpha}(h)\bigl(\mathrm{d}p^\alpha + B^\alpha_j(h,p)\,\eta^j\bigr) \wedge \eta^i$$

for some functions B^α_j. (*Torsion absorbable hypothesis*)

Then: Modulo diffeomorphism, the general real-analytic solution depends on s_q functions of q variables. Moreover, one can specify h and p arbitrarily at a point.

Remark: The proof is a straightforward modification of Cartan's proof in the case $r = 0$ (i.e., when there are no 'free derivatives' p^α).

Example: Riemannian 3-manifolds with const. Ricci eigenvalues.

For simplicity, assume the eigenvalues are all distinct, with $c_1 > c_2 > c_3$. Then there exist ω_i, ϕ_i so that $g = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2$ and

$$\begin{aligned} \mathrm{d}\omega_1 &= \phi_2\wedge\omega_3 - \phi_3\wedge\omega_2 & \mathrm{d}\phi_1 &= \phi_2\wedge\phi_3 + c_1\,\omega_2\wedge\omega_3 \\ \mathrm{d}\omega_2 &= \phi_3\wedge\omega_1 - \phi_1\wedge\omega_3 & \mathrm{d}\phi_2 &= \phi_3\wedge\phi_1 + c_2\,\omega_3\wedge\omega_1 \\ \mathrm{d}\omega_3 &= \phi_1\wedge\omega_2 - \phi_2\wedge\omega_1 & \mathrm{d}\phi_3 &= \phi_1\wedge\phi_2 + c_3\,\omega_1\wedge\omega_2 \end{aligned}$$

2^{nd} Bianchi implies that there are functions a_i and b_i so that

$$\begin{aligned} \phi_1 &= a_1\,\omega_1 + (c_1{-}c_3)\,b_3\,\omega_2 + (c_1{-}c_2)\,b_2\,\omega_3 \\ \phi_2 &= a_2\,\omega_2 + (c_2{-}c_1)\,b_1\,\omega_3 + (c_2{-}c_3)\,b_3\,\omega_1 \\ \phi_3 &= a_3\,\omega_3 + (c_3{-}c_2)\,b_2\,\omega_1 + (c_3{-}c_1)\,b_1\,\omega_2\,. \end{aligned}$$

$\mathrm{d}(\mathrm{d}\omega_i) = 0$, then yields 9 equations for $\mathrm{d}a_i$, $\mathrm{d}b_i$. These can be written in the form

$$\begin{aligned} \mathrm{d}a_i &= \bigl(A_{ij}(a,b) + p_{ij}\bigr)\,\omega_j \\ \mathrm{d}b_i &= \bigl(B_{ij}(a,b) + q_{ij}\bigr)\,\omega_j\,, \end{aligned}$$

where $q_{ii} = 0$ and $q_{ij} = -p_{jk}/(c_i{-}c_j)$ when (i,j,k) are distinct.

$$
\begin{aligned}
\mathrm{d}\omega_1 &= \phi_2\wedge\omega_3 - \phi_3\wedge\omega_2 &\qquad \phi_1 &= a_1\,\omega_1 + (c_1{-}c_3)\,b_3\,\omega_2 + (c_1{-}c_2)\,b_2\,\omega_3\\
\mathrm{d}\omega_2 &= \phi_3\wedge\omega_1 - \phi_1\wedge\omega_3 &\qquad \phi_2 &= a_2\,\omega_2 + (c_2{-}c_1)\,b_1\,\omega_3 + (c_2{-}c_3)\,b_3\,\omega_1\\
\mathrm{d}\omega_3 &= \phi_1\wedge\omega_2 - \phi_2\wedge\omega_1 &\qquad \phi_3 &= a_3\,\omega_3 + (c_3{-}c_2)\,b_2\,\omega_1 + (c_3{-}c_1)\,b_1\,\omega_2\,.
\end{aligned}
$$

$$
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\end{aligned}
$$

where $q_{ii}=0$ and $q_{ij} = -p_{jk}/(c_i{-}c_j)$ when (i,j,k) are distinct.

This defines an involutive tableau (in the p-variables) of rank $r=9$ and with characters $s_1=6$, $s_2=3$, and $s_3=0$. Cartan's criteria are satisfied, so the desired metrics depend on three functions of two variables.

A similar analysis applies when $c_1=c_2\neq c_3$, showing that such metrics depend on one function of two variables.

When $c_1=c_2=c_3$, the Cartan analysis gives the expected result that the solutions depend on a single constant.

General Holonomy. A torsion-free H-structure on M^n (where $H \subset \mathrm{GL}(\mathfrak{m})$ and $\dim(\mathfrak{m}) = n$) satisfies the first structure equation

$$\mathrm{d}\omega = -\phi \wedge \omega$$

where ϕ takes values in $\mathfrak{h} \subset \mathrm{GL}(\mathfrak{m})$ and the second structure equation

$$\mathrm{d}\phi = -\phi \wedge \phi + R(\omega \wedge \omega)$$

where R takes values in $K^0(\mathfrak{h})$, the space of curvature tensors. Second Bianchi becomes

$$\mathrm{d}R = -\phi \cdot R + R'(\omega)$$

where R' takes values in $K^1(\mathfrak{h}) \subset K^0(\mathfrak{h}) \otimes \mathfrak{m}^*$.

Theorem: For all groups H satisfying Berger's criteria *except* the exotic symplectic list, $K^1(\mathfrak{h})$ is an involutive tableau and the above equations satisfy Cartan's criteria.

Ex: For $G_2 \subset \mathrm{GL}(7, \mathbb{R})$, the tableau $K^1(\mathfrak{g}_2)$ has $s_6 = 6 > s_7 = 0$, so the general metric with G_2-holonomy depends on 6 functions of 6 variables.

Other examples.

1. (Cartan, 1926) The metrics in dimension 4 with holonomy SU(2) depend locally on 2 functions of 3 variables.

2. (Cartan-Einstein, 1929-32) Analysis of Einstein's proposed unified field theory via connections with torsion.

3. (Cartan, 1943) The Einstein-Weyl structures in dimension 3 depend on 4 functions of 2 variables.

4. (B—, 1987) The Ricci-solitons in dimension 3 depend on 2 functions of 2 variables. (More generally, $\mathrm{Ric}(g) = a(f)\,g + b(f)\,\mathrm{d}f^2 + c(f)\ \mathrm{Hess}_g(f)$.)

5. (B—, 2008) The solitons for the G_2-flow in dimension 7 depend on 16 functions of 6 variables.