Integrability of Second-Order Partial Differential Equations and the Geometry of GL(2)-Structures

by

Abraham D. Smith

Department of Mathematics
Duke University

Date: __________________________________

Approved:

____
Robert L. Bryant, Advisor

____
William Allard

____
Paul Aspinwall

____
Mark Stern

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2009

Abstract
(Mathematics)

Integrability of Second-Order Partial Differential Equations and the Geometry of GL(2)-Structures

by

Abraham D. Smith

Department of Mathematics
Duke University

Date: ______________________

Approved:

____
Robert L. Bryant, Advisor

____
William Allard

____
Paul Aspinwall

____
Mark Stern

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University
2009

Abstract

A $GL(2,\mathbb{R})$-structure on a smooth manifold of dimension $n+1$ corresponds to a distribution of non-degenerate rational normal cones over the manifold. Such a structure is called k-integrable if there exist many foliations by submanifolds of dimension k whose tangent spaces are spanned by vectors in the cones.

This structure was first studied by Bryant ($n = 3$ and $k = 2$). The work included here ($n = 4$ and $k = 2, 3$) was suggested by Ferapontov, *et al.*, who showed that the cases ($n = 4, k = 2$) and ($n = 4, k = 3$) can arise from integrability of second-order PDEs via hydrodynamic reductions.

Cartan–Kähler analysis for ($n = 4, k = 3$) leads to a complete classification of local structures into 55 equivalence classes determined by the value of an essential 9-dimensional representation of torsion for the $GL(2,\mathbb{R})$-structure. These classes are described by the factorization root-types of real binary octic polynomials. Each of these classes must arise from a PDE, but many of the PDEs remain to be identified.

Also, we study the local problem for $n \geq 5$ and $k = 2, 3$ and conjecture that similar classifications also exist for these cases; however, the most interesting results are essentially unique to degree 4. The approach is that of moving frames, using Cartan's method of equivalence, the Cartan–Kähler theorem, and Cartan's structure theorem.

Contents

List of Figures

List of Abbreviations and Symbols

Abbreviations

EDS	Exterior Differential System.
ODE	Ordinary Differential Equation.
PDE	Partial Differential Equation.

Symbols

Sorted by order of appearance, beginning in Chapter 1.

$\mathbf{J}$	The space of 1-jets over $\mathbb{R}^3$.
$\mathbf{T}$	The tangent space of a manifold.
$\mathcal{I}$	The differential ideal for an EDS.
Ω	The independence condition for an EDS.
$Gr_3(\mathbb{R}^6)$	The Grassmann manifold of 3-planes in $\mathbb{R}^6$.
M	A smooth or real-analytic manifold of dimension $n+1$.
$\bar{\mathbf{J}}$	The space of reduced 1-jets over $\mathbb{R}^3$.
$Sp(3,\mathbb{R})$	The symplectic subgroup of $GL(\mathbb{R}^6)$.
Γ	The space of sections of the indicated bundle.
Λ	The Lagrangian Grassmannian.
Λ^o	The open Lagrangian Grassmannian.
$\mathcal{C}$	The standard rational normal curve in $\mathcal{V}_n$.
$\mathbb{P}$	The projectivization of a vector space.

PGL	The projective general linear group.
$\mathbf{V}$	A distribution of Veronese cones over Λ^o.
$\mathbf{C}$	A distribution of rational normal cones over a manifold.
$\mathcal{F}$	The co-frame bundle of a manifold.
$GL(\mathcal{V}_n)$	The general linear group over $\mathcal{V}_n$.
$\mathfrak{g}$	The Lie algebra associated to a Lie group G (for any G).
d	The exterior derivative.
ω	The tautological 1-form of a principal bundle.
θ	The connection 1-form of a principal bundle.
T	The torsion of a G-structure. Eventually, the unique irreducible torsion of a $GL(2,\mathbb{R})$-structure.
∇	The covariant derivative operator, $\mathrm{d}+\theta$.
$\mathcal{O}$	The leaf-space for a closed system of structure equations.
$GL(2)$	The general linear group over a two-dimensional real vector space.
$\mathcal{V}_n$	The vector space of nth degree binary polynomials.
x	Lowest weight generator in $\mathcal{V}_1$.
y	Highest weight generator in $\mathcal{V}_1$.
φ	An $\mathfrak{sl}(2)$-valued 1-form of a $GL(2)$-structure.
λ	An $\mathbb{R}$-valued 1-form of a $GL(2)$-structure.
τ	The torsion of a tableau for an EDS.
Q	Quadratic term in T, from the Bianchi identity.
R	Curvature of a $GL(2)$-structure, from the Bianchi identity.
S	Covariant derivative of T, from Bianchi identity.

Acknowledgements

This dissertation, and my mathematical career in general, would not have been possible without the collective effort of many individuals. Foremost among them is my advisor, Robert Bryant. Always patient and generous with his limited time, Robert has taught me (and re-taught me) all that I understand about geometry. If only I could listen more carefully and think more precisely, then I would understand twice as much as I do today. I also wish to thank the members of my committee, Bill Allard, Paul Aspinwall, and Mark Stern, along with Hugh Bray and Les Saper, who frequently faced my malformed and ill-posed questions. It is a formidable task to make sense of another's confusion and direct him to the proper path.

Many post-docs and grad students at Duke contributed in a thousand ways to a stimulating mathematical and cultural environment. Thank you all for putting up with my idle banter! Several people outside of Duke have also been very helpful. This project would have been much more tedious without the Cartan package for Maple written and supported by Jeanne Clelland. I also owe thanks to Pawel Nurowski, Niky Kamran, and Dennis The for our various conversations, which convinced me that this research is both interesting and important.

Finishing a dissertation is not an intellectual achievement so much as a psychological battle. The preservation of my waning sanity was achieved only with the support of my lovely wife, Katie, and the help of Loki, who knows what life is for.

Introduction

In the first few decades of the twentieth century, Élie Cartan developed several general methods to analyze differential-geometric objects that arise by specifying a structure on the tangent bundle of a manifold. The method of the moving frame, wherein a frame is fixed that respects the specified structure, provides an efficient means to calculate quantities relating to the structure. The method of equivalence and Cartan's structure theorem allow one to determine the local freedom of such a structure. If additional restrictions are placed on the structure (such as the existence of special submanifolds), they are usually equivalent to an overdetermined system of partial differential equations; the Cartan–Kähler theorem and its corollaries supply efficient algorithms to determine the existence and abundance of solutions to these PDEs. For all of these methods, the differential calculus of exterior forms provides the most efficient and most aesthetically satisfying language. Collectively, these methods comprise the subject "Exterior Differential Systems." The canonical reference for this subject is [BCG$^+$91], written by many of the subject's main proponents during the end of the twentieth century.

Using these tools (and little else), we set forth to study the integrability of $GL(2,\mathbb{R})$-structures. To specify a $GL(2,\mathbb{R})$-structure is to specify a smooth field of rational normal cones on M. That is, a $GL(2,\mathbb{R})$-structure is a choice of non-degenerate $\mathbb{P}^1 \subset \mathbb{P}\mathbf{T}_pM$ that varies smoothly with $p \in M$. There are two notions of integrability for a $GL(2,\mathbb{R})$-structure over M, but they both correspond to the existence of surfaces $\Sigma \subset M$ such that $\mathbf{T}_p\Sigma$ is either spanned by or tangent to vectors in the rational normal cone. These definitions are developed in Chapter 1 and Chapter 3.

$GL(2,\mathbb{R})$-structures have arisen in three notable contexts. First, as part of a program to complete and correct Berger's list of Riemannian and torsion-free affine holonomy, Bryant sought in 1991 to construct manifolds of dimension 4 with torsion-free affine holonomy of $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$ [Bry91]. His approach naturally produces a $GL(2,\mathbb{R})$-structure on the tangent bundle, and Bryant showed that the existence of these holonomies is tied to integrability of this structure. At that time Bryant also noted a link between this $GL(2,\mathbb{R})$-structure and a path-geometric interpretation of fourth-order ODEs in a single variable, effectively continuing work of Chern and Cartan, [Che40] and [Car41]. This work on ODEs has continued, with many interesting results appearing recently in Europe from the perspective of either $GL(2,\mathbb{R})$ geometry or conformal $CO(3)$ geometry [Dou01] [DT06] [GN06] [CS07] [GN07] [Nur07] [BN07] [GN09]. Most relevant to our current study, a $GL(2,\mathbb{R})$-structure was recently discovered by Ferapontov and others in [FHK07]. There it is shown that certain classical PDEs of second order give rise to $GL(2,\mathbb{R})$-structures on naturally defined manifolds of dimension 5. Strikingly, the integrability conditions for the PDEs correspond exactly to the geometric notion of integrability for the $GL(2,\mathbb{R})$-structure.

This dissertation examines the notion of $GL(2,\mathbb{R})$ integrability that arises in this last case. The PDE context for the problem is outlined in Chapter 1, which mostly serves as a summary of the relevant results from [FHK07]. In Chapter 2 the aforementioned techniques of Cartan are summarized for later use. Notation is established in Chapter 3, where the relevant fiber bundles and representation theory are introduced and k-integrability of a degree-n $GL(2,\mathbb{R})$ structure over a manifold of dimension $n+1$ is defined. In Chapter 4 we restrict our attention to dimension 5 (degree 4), the case that arises in [FHK07]. Using Cartan's method of equivalence, we find a global coframing for a $GL(2,\mathbb{R})$-structure of degree 4 that is canonical but not torsion-free. Imposing the integrability conditions in Chapter 5 and Chapter 6, we use the Cartan–Kähler machinery to show that a particular irreducible 9-dimensional representation of torsion uniquely characterizes local 3-integrable $GL(2,\mathbb{R})$-structures of degree 4 that are also 2-integrable. Chapter 7 contains some simple technical observations regarding the algebraic behavior of this irreducible representation. Encountering extraordinary luck in Chapter 8, we are able to use those technical observations to completely classify $GL(2,\mathbb{R})$-structures of degree 4 that are both 2- and 3-integrable into 55 equivalence classes, corresponding to the factorization root-types of real binary octics. In some cases, the structure equations are integrated to give an example of each type in local coordinates. Chapter 9 completes the picture of integrability in degree 4, as we determine that all of the 55 classes are locally embeddable into the symplectic bundle over the Lagrangian Grassmannian; the existence of such an embedding indicates that the structure arises from a PDE as described in [FHK07]. In Chapters 10 and beyond, we use the same methods to examine $GL(2,\mathbb{R})$-structures in higher degrees. A global, canonical coframing is obtained, and some interesting integrability and classification results arise, but their meaning is not yet clear.

It is well-known that Cartan's methods can lead to computational madness, and it appears that $GL(2,\mathbb{R})$-structures are particularly afflicted. As with [Bry91] and [FHK07], the majority of results contained herein would be impossible but for the uncanny speed and accuracy of computer algebra software. The bulk of the results here were obtained using Maple versions 9, 9.5, 10, 11, and 12 for Linux and occasionally verified using Macaulay2 version 1.1. In order to provide reasonable evidence for my assertions, I have attempted to clearly lay out the nature of the computations necessary for each proof—most of them are "merely" determining solvability of linear systems—but the computational details are generally incomprehensible and thus uninformative. The most striking results regarding classification of 3-integrable structures are possible only because of the unique properties of the matrix J. The entries of this matrix are included in Appendix A; that data should allow the reader to verify many of the results simply by choosing a particular binary octic and applying elementary linear algebra.

In the context of integrable PDEs, this project has a broader purpose. Our overall understanding of integrability phenomena is extremely limited. Since integrability is essential to the study of PDEs in the physical sciences, this situation is singularly unsatisfying in contemporary mathematics. To a geometer the preferred outcome is to have a coordinate-invariant means to decide whether a PDE is integrable by studying the symmetries or invariants of an associated geometric structure. For the class of PDEs that are integrable by a three-parameter family of hydrodynamic reductions, this goal is nearly achieved by combining [FHK07] and this dissertation. Whether this classification can be extended to second-order PDEs involving lower derivatives is still unclear, but it is the next reasonable question to address.

1

Background

This chapter is a summary of the relevant PDE theory. This chapter begins by reviewing the correspondence between C^1 functions and graphs in jet space along with the naturally-arising symplectic structure. Additional consequences are noted in the case of second-order PDEs, and finally some results of [FHK07] are summarized to provide motivation.

1.1 Functions and Jets

Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function and that $\mathbb{R}^3$ has coordinates $\xi = (\xi^1, \xi^2, \xi^3)$. The function $u \in C^1(\mathbb{R}^3)$ gives rise to a section of the 1-jet space $\mathbf{J} = \mathbb{R}(\xi^1, \xi^2, \xi^3, z, p_1, p_2, p_3)$. This section may be regarded as a graph with coordinates $(\xi^1, \xi^2, \xi^3, u(\xi), u_1(\xi), u_2(\xi), u_3(\xi))$ where $u_i = \frac{\partial u}{\partial \xi^i}$.

Conversely, suppose $N \subset \mathbf{J}$ is a three-dimensional submanifold that projects onto $\mathbb{R}^3$. It is possible that N is the jet-graph of some C^1 function u; the necessary and sufficient condition can be described neatly in terms of an EDS: Consider the ideal $\mathcal{I}$ in the exterior algebra over $\mathbf{J}$ that is generated by $\{\mathrm{d}z - p_i \mathrm{d}\xi^i, \mathrm{d}p_i \wedge \mathrm{d}\xi^i\}$. (Note the use of the repeated-index summation convention, which is ubiquitous in this dissertation.) In the language of exterior differential systems, a tangent plane $\pi \in Gr(\mathbf{TJ})$ is called an *integral element* if $\psi|_\pi = 0$ for all $\psi \in \mathcal{I}$, and a submanifold N is called *integral to* $\mathcal{I}$ if $N^*(\psi) = 0$ for all $\psi \in \mathcal{I}$. This leads us to one of the most elementary observations in the subject of exterior differential systems [GS87].

Lemma 1.1. *Suppose N is a submanifold of $\mathbf{J}$ of dimension three. The following are equivalent:*

1. *N is locally the jet-graph of the function $u : \xi \mapsto N^*(z)$.*

2. *N is an integral submanifold of $\mathcal{I}$ and $N^*(\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3) \neq 0$.*

The non-degeneracy condition $N^*(\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3) \neq 0$ is needed so that the projection $N \to \mathbb{R}^3$ is a submersion. If this submersion is onto, then the function u is defined globally on $\mathbb{R}^3$, in which case N is actually a section of $\mathbf{J}$. A section satisfying the criteria of Lemma 1.1 is said to be *holonomic*.

Given a particular N and any $z_0 \in \mathbb{R}$, the translated submanifold $N + z_0$ is locally the graph of the translated function $u + z_0$. So elements of $C^1/\mathbb{R}$ (by addition of constants) correspond to submanifolds of $\overline{\mathbf{J}} = \mathbf{J}/(z) = \mathbb{R}(\xi^1, \xi^2, \xi^3, p_1, p_2, p_3)$ that are integral to the ideal $\overline{\mathcal{I}}$ spanned by $\{\mathrm{d}p_i \wedge \mathrm{d}\xi^i\}$ in the exterior algebra over $\overline{\mathbf{J}}$. Readers familiar with exterior differential systems should recognize this reduction as the contraction of $\mathcal{I}$ and $\mathbf{J}$ along the infinitesimal symmetry $\frac{\partial}{\partial z}$.

The global two-form $\sigma = \mathrm{d}p_i \wedge \mathrm{d}\xi^i$ gives the manifold $\overline{\mathbf{J}}$ a natural symplectic structure. In other words, there is a principal bundle over $\overline{\mathbf{J}}$ that has fiber $Sp(3, \mathbb{R}) = \{A \in GL(6, \mathbb{R}) : \sigma(Av, Aw) = \sigma(v, w)\ \forall v, w \in \mathbb{R}^6\}$; this group provides the infinitesimal automorphisms on $\overline{\mathbf{J}}$ that preserve the structure σ. Indeed, writing

$$\begin{pmatrix} \mathrm{d}\tilde{\xi} \\ \mathrm{d}\tilde{p} \end{pmatrix} = \begin{pmatrix} B & C \\ A & D \end{pmatrix} \begin{pmatrix} \mathrm{d}\xi \\ \mathrm{d}p \end{pmatrix} \tag{1.1}$$

with the condition that $\mathrm{d}\tilde{p}_i \wedge \mathrm{d}\tilde{\xi}^i = 0 = \mathrm{d}p_i \wedge \mathrm{d}\xi^i$ forces the block-matrix definition of $Sp(3,\mathbb{R})$: $BA = (BA)^T$, $CD = (CD)^T$, and $CA - BD = I$.

With a symplectic structure available, particular tangent planes are naturally distinguished:

Definition 1.2 (Lagrangian planes)**.** *The Lagrangian Grassmannian is*

$$\Lambda = \{\pi \in Gr_3(\mathbb{R}^6) : \sigma|_\pi = 0\}. \tag{1.2}$$

The Lagrangian Grassmann bundle over $\overline{\mathbf{J}}$ *is*

$$Gr_3(\mathbf{T}\overline{\mathbf{J}}, \sigma) = \{\pi \in Gr_3(\mathbf{T}\overline{\mathbf{J}}) \ : \ \sigma|_\pi = 0\} = \overline{\mathbf{J}} \times \Lambda. \tag{1.3}$$

The open Lagrangian Grassmannian is

$$\Lambda^o = \{\pi \in \Lambda : \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3|_\pi \neq 0\} \subset \Lambda. \tag{1.4}$$

The open Lagrangian Grassmann bundle over $\overline{\mathbf{J}}$ *is*

$$Gr_3^o(\mathbf{T}\overline{\mathbf{J}}, \sigma) = \{\pi \in Gr_3(\mathbf{T}\overline{\mathbf{J}}, \sigma) \ : \ \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3|_\pi \neq 0\} = \overline{\mathbf{J}} \times \Lambda^o. \tag{1.5}$$

By the definition of σ, the 6-dimensional manifold Λ is exactly the space of 3-planes integral to $\overline{\mathcal{I}}$ over a specified point in $\overline{\mathbf{J}}$. Moreover, $Sp(3,\mathbb{R})$ is a principal bundle over Λ with fiber

$$\mathcal{P} \cong \left\{ \begin{pmatrix} B & C \\ 0 & B^{-1} \end{pmatrix} : B \in GL(3,\mathbb{R}), C = C^T \right\},$$

as $\mathcal{P}$ is the subgroup of $Sp(3,\mathbb{R})$ that preserves a particular integral element. The bundle $Sp(3,\mathbb{R})$ restricts to Λ^o, an open set in Λ.

With these objects in hand, Lemma 1.1 can be refined in the following manner:

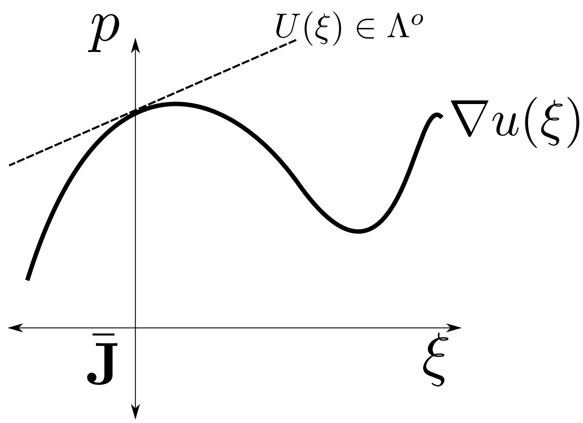

FIGURE 1.1: The reduced jet-graph of a C^2 function u and a tangent plane represented by U, the Hessian matrix of u.

Lemma 1.3. *Let $N \subset \overline{\mathbf{J}}$ be a submanifold of dimension three. The following are equivalent:*

1. *N is locally the reduced jet-graph of a function $u : \mathbb{R}^3 \to \mathbb{R}$ such that $u_i = N^*(p_i)$.*
2. *N is locally the graph of $\nabla u : \mathbb{R}^3 \to \mathbb{R}^3$ for some $u : \mathbb{R}^3 \to \mathbb{R}$.*
3. *$\mathbf{T}N \in \Gamma(Gr_3^o(\mathbf{T}\overline{\mathbf{J}}, \sigma))$*
4. *$\mathbf{T}_{(\xi,p)}N \in \Lambda^o$ for all $(\xi, p) \in N$.*

Lemma 1.3 states that Λ^o is exactly the set of integral elements on which the desired non-degeneracy condition, $\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3 \neq 0$, holds. An element $\pi \in \Lambda^o$ may be uniquely defined by a basis $\{X_1, X_2, X_3\}$ where $X_i = \frac{\partial}{\partial \xi^i} + U_{ij} \frac{\partial}{\partial p_j}$. Moreover, since π is integral to $\overline{\mathcal{I}}$, we may compute $0 = \sum_i \mathrm{d}\xi^i \wedge \mathrm{d}p_i(X_j, X_k) = U_{jk} - U_{kj}$. That is, on $\pi \in \Lambda^o$ we may write $\mathrm{d}p_i = U_{ij}\mathrm{d}\xi^j$, and the symmetric matrix U uniquely defines π. In other words, if $u \in C^2$ is a function corresponding to N where $u(\xi) = z|_N(\xi)$ and $u_i(\xi) = p_i|_N(\xi)$, then the matrix U is the Hessian of u: $U_{ij}(\xi) = \frac{\partial^2}{\partial \xi^i \partial \xi^j} u(\xi)$. See Figure 1.1. Therefore, $U : \Lambda^o \to \mathrm{Sym}^2(\mathbb{R}^3)$ is a diffeomorphism from the space of non-degenerate integral elements to the space of symmetric matrices.

1.2 The Rational Normal and Veronese Varieties

For future application we now briefly recall some elementary structures from algebraic geometry. An excellent reference for this material is [Har95], particularly Chapters 1, 2, and 18. For concreteness and direct application to our purpose, we fix the dimensions so that the target space has projective dimension 5; however all of these definitions extend easily to arbitrary dimension.

Recall that a variety is said to be "non-degenerate" if no proper linear subspace contains it. This condition is sometimes called "normal," but this usage conflicts with a more modern and more common usage that is related to the local ring of a variety. Nonetheless, the archaic usage it lends its name to a fundamental object in algebraic geometry, the rational normal curve. The *rational normal curve* is the image of the regular map $\mathbb{P}(\mathbb{R}^2) \to \mathbb{P}(\mathbb{R}^6)$ given by

$$[a, b] \mapsto [a^5, a^4b, a^3b^2, a^2b^3, ab^4, b^5]. \tag{1.6}$$

The de-projectivized version in $\mathbb{R}^6$ is called the *rational normal cone*, denoted $\mathcal{C}$. We also use the term "rational normal cone" to refer to any cone arising from this one by a projective automorphism of $\mathbb{R}^6$; in particular, we apply a diagonal basis change to $\mathbb{R}^6$ from Chapter 3 onwards. This definition is unambiguous, as all non-degenerate curves of degree n in $\mathbb{P}(\mathbb{R}^{n+1})$ are $PGL(\mathbb{R}^{n+1})$-congruent. For our purposes, the key fact about rational normal cones is Lemma 1.4, which arises from the simple fact that $PGL(2, \mathbb{R})$ is the automorphism group of $\mathbb{P}(\mathbb{R}^2)$ [Har95, Example 10.8].

Lemma 1.4. *The symmetry group of $\mathcal{C} \subset \mathbb{R}^{n+1}$ is $GL(2, \mathbb{R}) \subset GL(n+1, \mathbb{R})$.*

Of course, the symmetry group $GL(2, \mathbb{R})$ of $\mathcal{C}$ is not any of the trivial block-diagonal embeddings of $GL(2, \mathbb{R})$ into $GL(n+1, \mathbb{R})$, since $\mathcal{C}$ is non-degenerate.

Given a rational normal cone $\mathcal{C}$, a k-dimensional linear subspace $L \subset \mathbb{R}^6$ is called *k-secant* if $L \cap \mathcal{C}$ is a set of k distinct lines, which therefore span L. See

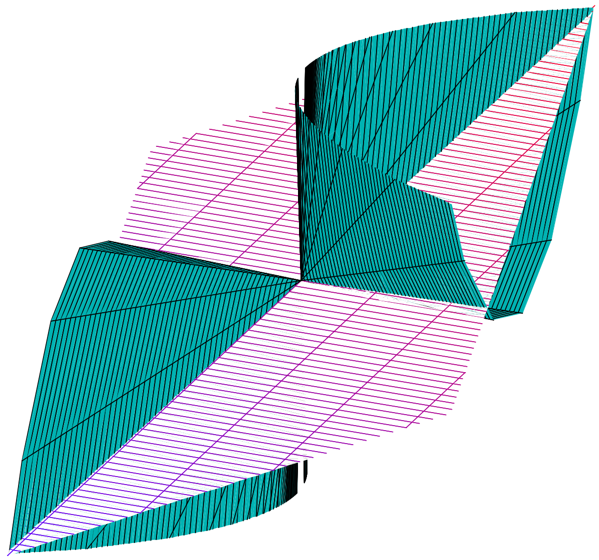

FIGURE 1.2: A rational normal cone in $\mathbb{R}^3$ with a bi-secant plane.

Figure 1.2. Note that this terminology is related to but distinct from the secant variety introduced in [Har95]. We are particularly interested in the case $k = 2$ (*bi-secant*) and $k = 3$ (*tri-secant*). The k-secant condition on $Gr_k(\mathcal{V}_n)$ is locally closed for $k < n$, but it is open for $k = n$ (that is, for subspaces that are co-dimension one in $\mathcal{V}_n$).

The *Veronese variety* generalizes the rational normal curve. For our purposes, it is defined as the image of the regular map $\mathbb{P}(\mathbb{R}^3) \to \mathbb{P}(\mathrm{Sym}^2(\mathbb{R}^3))$ given by

$$[Z_1, Z_2, Z_3] \mapsto \begin{bmatrix} Z_1Z_1 & Z_1Z_2 & Z_1Z_3 \\ Z_2Z_1 & Z_2Z_2 & Z_2Z_3 \\ Z_3Z_1 & Z_3Z_2 & Z_3Z_3 \end{bmatrix} \tag{1.7}$$

The de-projectivized version of the Veronese variety is the *Veronese cone*, and it can also be described in a coordinate-free manner as the set of matrices in $\mathrm{Sym}^2(\mathbb{R}^3)$ with rank at most 1. In the complex case, it is easy to check that the intersection of a generic hyperplane with the 3-dimensional Veronese cone produces a 2-dimensional rational normal cone in $\mathbb{C}^6 = \mathrm{Sym}^2(\mathbb{C}^3)$. We need the real case, which takes a little detail to describe accurately. Consider an intersection of a hyperplane with with the

Veronese cone. This intersection is given by the polynomial equation in Z_1, Z_2, Z_3 such as

$$a_{11}(Z_1)^2 + a_{12}Z_1Z_2 + a_{13}Z_1Z_3 + a_{22}(Z_2)^2 + a_{23}Z_2Z_3 + a_{33}(Z_3)^2 = 0. \tag{1.8}$$

Depending on the coefficients a_{ij}, Equation (1.8) may or may not have real solutions. If Equation (1.8) has real solutions, then the solution in $\mathbb{P}(\mathbb{R}^3)$ is a real quadric surface. This quadric may or may not be degenerate. The existence of real non-degenerate solutions is an open condition on the hyperplane in the topology of $Gr_3(\mathbb{R}^6)$. If this condition is satisfied, we say that the hyperplane defining $\{a_{ij}\}$ is *pure*. The exact algebraic condition for purity need not concern us and can be found in many places such as [ZKR96]. The relevant fact is that a pure hyperplane in $\mathrm{Sym}^2(\mathbb{R}^3)$ intersects the Veronese cone in a rational normal cone, and every rational normal cone in $\mathrm{Sym}^2(\mathbb{R}^3)$ can be written this way.

Returning to the symplectic jet-space geometry of the previous section, there is a natural and $Sp(3,\mathbb{R})$-invariant distribution, $\mathbf{V}$, of Veronese cones over Λ^o with fiber

$$\mathbf{V}_\pi = \{\varpi \in \mathbf{T}_\pi\Lambda^o : \operatorname{rank} \mathrm{d}U_\pi(\varpi) \leq 1\}. \tag{1.9}$$

1.3 Motivation via Second-Order PDEs

Consider now a partial differential equation of the form

$$F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0 \tag{1.10}$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function and $u_{ij} = \frac{\partial^2}{\partial\xi^i\partial\xi^j}u = u_{ji}$. In the context of Lemma 1.3, $F(U) = 0$ is a closed condition defining a subset of Λ^o.

Suppose $\pi \in \Lambda^o$ with $F(\pi) = 0$, and suppose additionally that the non-degeneracy condition $\mathrm{d}F(\pi) \neq 0$ holds. By the implicit function theorem, there is a 5-dimensional manifold M through π that is the intersection of $F^{-1}(0)$ with an open set in Λ^o.

Hence, we can informally state that $F(U) = 0$ locally defines a 5-dimensional submanifold $M \subset \Lambda^o$ near generic $\pi \in F^{-1}(0)$. If $\mathbf{T}_\pi M \subset \mathbf{T}_\pi \Lambda^o$ is pure, then it intersects the Veronese variety $\mathbf{V}_\pi \subset \mathbf{T}_\pi \Lambda^o$ in a 2-dimensional rational normal cone $\mathbf{C}_p$. This discussion may be summarized as follows:

Theorem 1.5. *Let $F : \Lambda^o \to \mathbb{R}$ be a smooth function, and suppose $\pi \in \Lambda^o$ such that $F(\pi) = 0$, $\mathrm{d}F(\pi) \neq 0$, and $\ker(\mathrm{d}F(\pi))$ is pure as a hyperplane in $\mathbf{T}_\pi \Lambda^o = \mathrm{Sym}^2(\mathbb{R}^3)$. Then there is an open 5-dimensional submanifold $M \subset \Lambda^o$ defined by $F|_M = 0$ in a neighborhood of π, and M admits a distribution $\mathbf{C}$ of rational normal cones. Equivalently, M admits a principal right $GL(2, \mathbb{R})$ bundle $B \to M$ that is a reduction of the co-frame bundle over M to the stabilizer of $\mathbf{C}$.*

The statement regarding $GL(2, \mathbb{R})$ bundles follows simply because the stabilizer group of $\mathbf{C}_p$ is exactly $GL(2, \mathbb{R})$, as in Lemma 1.4. This equivalence between $GL(2, \mathbb{R})$-structures and fields of rational normal cones is explicitly verified after notation is fixed for the co-frame bundle of M in Chapter 3.

A distribution of cones yields a notion of integrability by foliation.

Definition 1.6 (k-Integrability)**.** *A submanifold N is also called* k-secant *if $\mathbf{T}_q N$ is a k-secant subspace for all $q \in N$. A principal right $GL(2, \mathbb{R})$ bundle (or the associated distribution of cones $\mathbf{C}$) over a manifold M is called* k-integrable *if through every $p \in M$ and every k-secant subspace $L \subset \mathbf{T}_p M$, there passes a k-secant submanifold N such that $\mathbf{T}_p N = L$.*

[FHK07] demonstrates an intriguing relationship between these foliations and a certain class of PDE. We now summarize their main result, which serves as motivation for this project.

Theorem 1.7 ([FHK07]). *Consider a PDE of the form $F(u_{ij}) = 0$, $1 \leq i, j \leq 3$, and a corresponding 5-dimensional manifold $M \subset \Lambda^o$ defined by $F^{-1}(0)$. Then*

1. *The natural $GL(2, \mathbb{R})$-structure over M is 2-integrable.*

2. *The PDE F is integrable via hydrodynamic reductions if and only if the natural $GL(2, \mathbb{R})$-structure over M is 3-integrable.*

PDEs that are integrable by means of hydrodynamic reductions have been extensively studied in a variety of physical contexts, ranging from general relativity to the dynamics of gas chromatography [Kod88] [KG89] [Tsa90] [GT96] [Tsa93] [Tsa00] [BLP03] [Pav03] [FO07]. A complete discussion of this integrability theory and the applications of hydrodynamic reductions is beyond the scope of this dissertation; [FHK07] provides sources for the interested reader. [FK03] serves as a collection of interesting examples, such as $u_{yy} = u_{xt} + \frac{1}{2}u_{xx}^2$ (the first flow of dispersionless Kadomtsev–Petviashvili equation), $u_{xx} + u_{yy} = e^{u_{tt}}$ (the Boyer-Finley equation) and many more, including various integrable hierarchies.

One hopes to integrate these PDEs by stipulating that the solution function u (and its derivatives) may be written as $u(R^1, \ldots, R^k)$ for an *a priori* unknown number of functions $R^1(\xi), \ldots, R^k(\xi)$ whose derivatives admit the "commuting" relations

$$\frac{\partial}{\partial \xi^2} R^i = \rho_2^i(R) \frac{\partial}{\partial \xi^1} R^i, \qquad \frac{\partial}{\partial \xi^3} R^i = \rho_3^i(R) \frac{\partial}{\partial \xi^1} R^i \tag{1.11}$$

which also imply

$$\frac{1}{\rho_2^i - \rho_2^j} \frac{\partial \rho_2^i}{\partial R^j} = \frac{1}{\rho_3^i - \rho_3^j} \frac{\partial \rho_3^i}{\partial R^j}, \quad \forall i \neq j. \tag{1.12}$$

The system of PDEs in Equations (1.11) and (1.12) is called a *k-component system of hydrodynamic type*. The functions R^i are sometimes called Riemann invariants. Solving a PDE using this assumption is called a *k-parameter hydrodynamic reduction* of the PDE. A PDE is called *integrable via hydrodynamic reductions* if, for any k, it admits infinitely many k-parameter hydrodynamic reductions that are parametrized by k functions of one variable. For a PDE on $u : \mathbb{R}^3 \to \mathbb{R}$, the existence of k-parameter hydrodynamic reductions is trivial for $k > 3$. Hence, the existence of the 3-parameter hydrodynamic reductions is sufficient to provide integrability of the PDE.

This process seems rather *ad hoc*, but it applies to a wealth of examples, as demonstrated in the above references. Indeed, [FHK07] also shows that the action of $Sp(3)$, which rewrites $F(U) = 0$ in terms of new coordinates on Λ^o, produces an open orbit of such PDEs. (This result is verified in a more precise manner in Chapter 9.)

From the geometric perspective, it is clear that describing the local flexibility of 2- and 3-integrable $GL(2,\mathbb{R})$-structures over 5-manifolds would be useful for understanding the geometric content of this method, and a complete local classification of 2- and 3-integrable structures would be even better. These are the primary goals of this dissertation. After they are achieved in Chapters 4 through 9, our attention turns to the generalization of k-integrable $GL(2,\mathbb{R})$-structures over M^{n+1} for arbitrary n and k.

2

Techniques of Cartan

As mentioned in the Introduction, the results demonstrated in this dissertation are proven using Cartan's method of moving frames, Cartan's method of equivalence, the Cartan–Kähler theorem, and Cartan's generalization of Lie's third fundamental theorem, which is called "Cartan's structure theorem" hereafter.

The method of the moving frame and the Cartan–Kähler theorem are now familiar to a wide audience, and excellent textbooks are available—notably [BCG+91] and [IL03]. These topics are not reviewed here, as a reasonable overview would undoubtedly expand to a text unto itself. The primary technique from Cartan–Kähler theory used in this project is Cartan's test for involutivity of linear Pfaffian systems.

The method of equivalence enjoys well-written coverage in [Gar89], [IL03], and [BGG03]. These references seem only known to a fairly small circle of researchers; however, the method is generally understood in some form by many differential geometers, often in the related contexts of affine connections or gauge groups.

Unfortunately, beyond Cartan's original work in [Car04], complete expositions of Cartan's structure theorem appear non-existent. Though summarizing comments are made in literature such as [Bry01], there does not appear to be a standard review of the complete classical version of this theorem. A modern descendant of the structure theorem, interpreted as integration of Lie algebroids to foliations by topological groupoids, has finally been established in [CF03], and there is a rich study of algebroids, groupoids, and pseudo-groups that can now subsume most of Cartan's original theorem.

This chapter outlines these two techniques of Cartan, both to sketch the important ideas for uninitiated readers and to fix terminology used in the main results of this project.

2.1 Tautological Forms and Connections

In this chapter, M is an m-dimensional manifold and $V = \mathbb{R}^m$. The *co-frame bundle* $\mathcal{F}$ over M is the bundle whose fiber is $\mathcal{F}_p = \{u_p : \mathbf{T}_pM \xrightarrow{\sim} V\}$, the set of all vector-space isomorphisms from the tangent space to V. We can write any element of $\mathcal{F}$ in components as $u = (u^i)$, and it is important to note that $u^1 \wedge \cdots \wedge u^m \neq 0$, since u is an isomorphism on each fiber.

Note that there is a right action of $GL(V)$ by $u \cdot g = g^{-1} \circ u$ that acts smoothly, simply and transitively on the fibers. Hence, $\mathcal{F}(M)$ is a principal right $GL(V)$ bundle, as shown in the following diagram:

$$\begin{array}{ccc} GL(V) \longrightarrow \mathcal{F}(M) & \ni & u : \mathbf{T}_pM \to V \\ \Big\downarrow \pi & & \Big\downarrow \\ M & \ni & p = \pi(u) \end{array} \tag{2.1}$$

For any $u \in \mathcal{F}$, there is a *tautological* 1*-form* $\omega_u : \mathbf{T}_u\mathcal{F} \to V$, defined as $\omega \in \Gamma(\mathbf{T}^*\mathcal{F} \otimes V)$ such that $\omega_u(z) = u(\pi_*(z))$ for any $z \in \mathbf{T}_u\mathcal{F}$. Using u to also refer to a section $u : M \to \mathcal{F}$, the following lemma gives ω a nice self-replication property on the sections.

Lemma 2.1. *$u^*(\omega) = u$ and $\omega^1 \wedge \cdots \wedge \omega^m \neq 0$.*

Proof. Compute pointwise as follows. Suppose $\pi(u) = p$, and let $v \in \mathbf{T}_pM$ be arbitrary. Then $u^*(\omega_u)_p(v) = \omega_u(u_*(v)) = u_p(\pi_*(u_*(v))) = u_p(v)$. The second statement follows trivially from the first and the earlier observation that u_p is a vector-space isomorphism. $\square$

To compute the derivative of ω, it is convenient to work locally. Let $U \subseteq M$ be an open set such that $\mathcal{F}(U)$ is trivial. Fix a local section $u \in \Gamma(\mathcal{F}(U))$. This section allows us to define a local trivialization $H : U \times GL(V) \to \mathcal{F}(U)$ by $H(p, g) = g^{-1}u_p$. By the previous lemma, $H^*(\omega) = g^{-1}u$.

Lemma 2.2 (Cartan's first structure equation)**.** *Let ω be the tautological form of a co-frame bundle $\mathcal{F}$. Then for some $\theta \in \Gamma(\mathbf{T}^*\mathcal{F} \otimes \mathfrak{gl}(V))$ and some $T : \mathcal{F} \to V \otimes \wedge^2 V^*$,*

$$\mathrm{d}\omega = -\theta \wedge \omega + T(\omega \wedge \omega). \tag{2.2}$$

Proof. We compute locally. Let $u \in \Gamma(\mathcal{F}(U))$ where $\mathcal{F}(U)$ has a specified local trivialization H as above. Note that u is basic on $U \times GL(V)$, so $\mathrm{d}u$ is semi-basic. Hence $\mathrm{d}u = Cu \wedge u$ for some $C : U \to V \otimes \wedge^2 V^*$, as the components of u span $\mathbf{T}^*U$. Then at $(p, g) \in U \times GL(V)$, we compute

$$\begin{aligned} H^*(\mathrm{d}\omega) &= \mathrm{d}H^*(\omega) = \mathrm{d}(g^{-1}u) \\ &= -g^{-1}\mathrm{d}gg^{-1} \wedge u + g^{-1}Cu \wedge u \\ &= -g^{-1}\mathrm{d}g \wedge H^*(\omega) + g^{-1}C(gg^{-1}u) \wedge (gg^{-1}u) \\ &= -g^{-1}\mathrm{d}g \wedge H^*(\omega) + g^{-1}CgH^*(\omega) \wedge gH^*(\omega). \end{aligned} \tag{2.3}$$

One may choose $\theta \in \Gamma(\mathbf{T}^*\mathcal{F} \otimes \mathfrak{gl}(V))$ such that $H^*(\theta) = g^{-1}\mathrm{d}g$ and $T = g^{-1}C(g, g) : \mathcal{F}(U) \to V \otimes \wedge^2 V^*$ satisfy Equation (2.2). $\square$

Note that θ and T depend on the splitting of $\mathbf{T}\mathcal{F}(U)$ determined by the local trivialization H. Even so, they are not uniquely defined. Such a 1-form θ is called a *connection* of the co-frame bundle $\mathcal{F}$, and θ pulls back to the fiber as the left-invariant Maurer–Cartan form on $GL(V)$. The function T is called the *torsion*, and its relationship to θ is crucial in what follows.

The notion of a connection is intimately related to the notion of a covariant derivative of a vector bundle. Suppose $\nabla : \Gamma(\mathbf{T}M) \to \Gamma(\mathbf{T}M \otimes \mathbf{T}^*M)$ is a covariant derivative on $\mathbf{T}M$. For a specified frame $\{e_i\}$ on M, the covariant derivative may be written $\nabla : e_j \mapsto A^i_j \otimes e_i$, where $A^i_j \in \Gamma(\mathbf{T}^*M \otimes V)$ is called the *affine connection*. For any $v = v^i e_i \in \Gamma(\mathbf{T}M)$, $\nabla(v) = \mathrm{d}(v^i)\otimes e_i + v^i A^j_i \otimes e_j = (\mathrm{d}+A)v$. If $(u^j) : M \to \mathcal{F}$ is the co-framing dual to (e_j), then $u^*(\theta^i_j) = A^i_j$. That is, the affine connection and the connection on the co-frame bundle are identical once a particular framing or co-framing has been chosen. The operator $\mathrm{d} + \theta$ is well-defined on the principal bundle $\mathcal{F}$ and corresponds to $\nabla = \mathrm{d} + A$ on the vector bundle $\mathbf{T}M$. In the context of $\mathcal{F}$, one often writes ∇ to mean $\mathrm{d} + \theta$. Confusion between these two closely related ∇'s is easily avoided by context; for our current purposes, ∇ is always the operator $\mathrm{d} + \theta$ on $\mathcal{F}$.

We now study the canonical differential relations that exist for a connection.

Lemma 2.3 (First and Second Bianchi identities)**.** $\nabla(\theta) \wedge \omega = \nabla(T(\omega \wedge \omega))$ *and* $\mathrm{d}\nabla(\theta) = \nabla(\theta) \wedge \theta - \theta \wedge \nabla(\theta)$.

Proof. The first statement is simply $\mathrm{d}^2\omega = 0$, computed using Equation (2.2). The second statement is a direct computation. □

Definition 2.4 (G-structure)**.** *Let G be a Lie subgroup of $GL(V)$. A G-structure on M is a G-subbundle of the co-frame bundle $\mathcal{F}$. More explicitly, a G-structure on M is a smooth submanifold $B \subset \mathcal{F}(M)$ such that $\pi : B \to M$ is a submersion and each fiber B_p is a G-orbit under the G-actions in $\mathcal{F}$.*

Obviously, $\mathcal{F}$ is just the naturally-defined $GL(V)$-structure on a smooth manifold M. The tautological form ω pulls back to any G-structure B, and with it come the first structure equation and some notion of connection and torsion. A connection for a G-structure B is of the form $\theta \in \Gamma(\mathbf{T}^*B \otimes \mathfrak{g})$ and is obtained from the vertical component of the pull-back of the connection on $\mathcal{F}(M)$.

2.2 The Method of Equivalence

We want to determine when two G-structures are identical up to a diffeomorphism. Informally, they must have the same smooth bundle structure, but it is also important that diffeomorphism is G-equivariant so that the action by G is respected.

Let B be a G-structure over M. Suppose $f : M \to \hat{M}$ is a diffeomorphism. There is an induced map $f^1 : B \to \mathcal{F}(\hat{M})$ given by $f^1 : u \mapsto u \circ (f_*)^{-1}$, as shown in the following diagram.

$$\begin{tikzcd} \mathbf{T}_pM \arrow[r, "f_*"] \arrow[d, "u"'] & \mathbf{T}_{f(p)}\hat{M} \arrow[d, "f^1(u)=u\circ(f_*)^{-1}"] \\ V \arrow[r, "\simeq"] & V \end{tikzcd} \tag{2.4}$$

Note that f^1 is canonical since $(f \circ h)^1 = f^1 \circ h^1$ for diffeomorphisms f and h. Also, $f^1(u \cdot g) = f^1(u) \cdot g$ for all $g \in G$. This leads us to the obvious definition of G-equivalence:

Definition 2.5 (Equivalence of G-structures)**.** *Two G-structures $B \subseteq \mathcal{F}(M)$ and $\hat{B} \subseteq \mathcal{F}(\hat{M})$ are said to be G-*equivalent *if there exists a diffeomorphism $f : M \to \hat{M}$ such that $f^1(B) = \hat{B}$.*

Hence, we can determine whether two G-structures are G-equivalent by producing (or excluding the existence of) a map $B \to \hat{B}$ satisfying appropriate conditions.

Theorem 2.6. *If $f : M \to \hat{M}$ is a diffeomorphism, then $(f^1)^*(\hat{\omega}) = \omega$. Conversely, if $F : B \to \hat{B}$ is a G-equivariant diffeomorphism such that $F^*(\hat{\omega}) = \omega$, then there* *exists a unique diffeomorphism $f : M \to \hat{M}$ such that $F = f^1$.*

Proof. Suppose that $f : M \to \hat{M}$ is a diffeomorphism. Then for any section u of B, we obtain a section $\hat{u} = f^1(u)$. Let $v \in \mathbf{T}_u B$. Then we compute

$$(f^1)^*(\hat{\omega}_{\hat{u}})_u(v) = \hat{\omega}_{\hat{u}}((f^1)_*(v)) = \hat{u}(\hat{\pi}_* \circ (f^1)_*(v)) = u((f_*)^{-1} \circ \hat{\pi}_* \circ (f^1)_*(v)). \quad (2.5)$$

However, $\hat{\pi}_* \circ (f^1)_* = f_* \circ \pi$, so $(f^1)^*(\hat{\omega}_{\hat{u}})_u(v) = u(\pi_*(v)) = \omega_u(v)$.

Conversely, assume we have an $F : B \to \hat{B}$ with the desired properties. Choose a section $u : M \to B$, and define $f = \hat{\pi} \circ F \circ u$. Note that f is a diffeomorphism $M \to \hat{M}$, as $f_* = \hat{\pi}_* \circ F_* \circ u_*$ has maximal rank and F is a diffeomorphism. In fact f is independent of the choice of u (hence unique): any other choice $\underline{u} = u \cdot g = g^{-1}u$ produces

$$\underline{f} = \hat{\pi} \circ F \circ \underline{u} = \hat{\pi} \circ F \circ (g^{-1}u) = \hat{\pi} \circ g^{-1}F(u) = \hat{\pi} \circ F(u) = f, \quad (2.6)$$

since $g^{-1}F(u)$ and $F(u)$ are in the same fiber. □

Theorem 2.6 provides the perfect corollary necessary to express G-equivalence as an EDS, which is very fruitful here. Recognizing that a map $B \to \hat{B}$ is a graph in $B \times \hat{B}$, we seek an EDS whose integral manifolds imply the existence of graphs in $B \times \hat{B}$. In particular, let $\mathcal{I}$ be the EDS on $B \times \hat{B}$ that is differentially generated by $\omega - \hat{\omega}$ with independence condition $\Omega = \omega^1 \wedge \cdots \wedge \omega^m \neq 0$.

Corollary 2.7. *An m-dimensional integral manifold of $(\mathcal{I}, \Omega)$ containing $(b, \hat{b})$ exists if and only if there are open neighborhoods $N \ni b$ and $\hat{N} \ni \hat{b}$ such that N and $\hat{N}$ are G-equivalent via a G-equivariant diffeomorphism that maps b to $\hat{b}$.*

The independence condition $\Omega \neq 0$ guarantees that the projection from the integral manifold to B (or $\hat{B}$) is a submersion. If this submersion is onto, then the G-equivalence is global. Of course, the usual EDS approach only provides local existence of integral manifolds, so we can only conclude local G-equivalence near a specified point. Also, if integration of the ideal requires Cartan–Kähler techniques, then the analytic category is required.

In any case, to solve the equivalence problem, we could study the ideal differentially generated by $\omega - \hat{\omega}$. Cartan's first structure equation tells us that the derivative of this generator is

$$\mathrm{d}(\omega - \hat{\omega}) = -(\theta - \hat{\theta}) \wedge \omega + (T - \hat{T})\ \omega \wedge \omega. \tag{2.7}$$

By the independence condition $\Omega \neq 0$, solutions exist only if we can absorb the torsion term into the connection term. Once $(T - \hat{T})$ is absorbed, we must analyze the tableau given by the connection term, $(\theta - \hat{\theta})$, to determine integrability of the EDS.

To simplify this analysis, instead suppose we had a canonical form for the connections such that the torsions were minimized in the sense that a maximum number of irreducible G-representations of T and $\hat{T}$ are absorbed into θ and $\hat{\theta}$, respectively. If θ, $\hat{\theta}$, T, and $\hat{T}$ were rewritten canonically in this way, we could conclude that two G-structures are not G-equivalent whenever $T \neq \hat{T}$. Hence, we aim to solve the equivalence problem by studying the freedom available in the connection of any particular G-structure B over M, thus determining a canonical form for θ and T.

Fix a G-structure B over M with tautological form ω, connection $\theta \in \Gamma(\mathbf{T}^*B \otimes \mathfrak{g})$, and torsion $T : B \to V \otimes \wedge^2 V^*$. Using indices to clarify matrix and summation operations, the first structure equation is

$$\mathrm{d}\omega^i = -\theta^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk}\ \omega^j \wedge \omega^k. \tag{2.8}$$

Suppose ${}^*\theta^i_j$ is another connection on B, so ${}^*\theta^i_j = \theta^i_j + P^i_{jk}\ \omega^k$ for some $P : B \to \mathfrak{g} \otimes V^*$. Let δ be the composition $\mathfrak{g} \otimes V^* \hookrightarrow V \otimes V^* \otimes V^* \to V \otimes \wedge^2 V^*$, which computes the skew part of P. The resulting change in torsion is given by

$$\begin{aligned}
\mathrm{d}\omega^i + {}^*\theta^i_j \wedge \omega^j &= \mathrm{d}\omega^i + (\theta^i_j + P^i_{jk}\omega^k) \wedge \omega^j \\
&= \frac{1}{2}T^i_{jk}\,\omega^j \wedge \omega^k + P^i_{jk}\,\omega^k \wedge \omega^j \\
&= \frac{1}{2}\big(T^i_{jk} - \delta P^i_{jk}\big)\omega^j \wedge \omega^k.
\end{aligned} \tag{2.9}$$

Hence, a change of $\theta + P\omega$ in the connection results in a change of $T - \delta P$ in the torsion.

The canonical form of a connection is revealed by using the map δ and some linear algebra. Let $\mathfrak{g}^{(1)} = \ker\delta$ and $H^{0,2}(\mathfrak{g}) = \operatorname{coker}\delta$, as represented in this exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g}^{(1)} \arrow[r] & \mathfrak{g}\otimes V^* \arrow[r, "\delta"] & V\otimes\wedge^2 V^* \arrow[r] & H^{0,2}(\mathfrak{g}) \arrow[r] & 0 \\
 & & & B \arrow[ul, "P"] \arrow[u, "T"'] \arrow[ur, "{[T]}"'] & &
\end{tikzcd} \tag{2.10}$$

The unabsorbable torsion of a canonically-written connection takes values in $H^{0,2}(\mathfrak{g})$, and the change of connection providing any particular unabsorbable torsion is unique if and only if $\mathfrak{g}^{(1)} = 0$. Generally if $\mathfrak{g}^{(1)}$ fails to vanish, the system must be prolonged to $\mathbf{T}B$; however in the situations encountered in Chapters 4, 11, and 10, $\mathfrak{g}^{(1)} = 0$ so this discussion is sufficient.

Readers should be familiar with the example of $SO(m)$-structures, where $\mathfrak{so}(m)^{(1)} = 0$ and $H^{0,2}(\mathfrak{so}(m)) = 0$; each $SO(m)$-structure admits a unique torsion-free connection, which is called the Levi-Cevita connection. Unlike a $SO(m)$-structure, which admits a *unique* canonical connection, our forthcoming structure admits *several* canonical connections, as seen in Theorem 4.1 and Theorem 10.1.

2.3 The Structure Theorem

Cartan's structure theorem studies the integrability of vector fields that satisfy certain bracket properties. This theorem is necessary to "integrate" the structure equations of G-structures (after the method of equivalence has reduced the torsion) and "count" the possible G-structures up to local G-equivariant diffeomorphism.

This section is essentially a detailed exposition of [Bry01, Appendix A], which is in turn a summary of certain results from [Car04]. Following Bryant, theorems are stated from the perspective of co-frames and structure equations, which is similar in manner to Cartan's foundational EDS work. This formulation is both reasonably straightforward and directly applicable to our forthcoming structure, Equation (6.11). A modern generalization of these results relies on the notions of Lie algebroids and Lie groupoids. An excellent summary of this modern theory is [Mac05], which supersedes the classic [Mac95], and like its predecessor contains an amazingly thorough bibliography with historical notes. The capstone result appears in [CF03], and a detailed exposition appears in [FC06].

For context and generalization, recall Lie's third fundamental theorem, which states that any putative structure equations that satisfy the conditions of a Lie bracket are realized by a unique local Lie group whose Lie algebra has those structure equations.

Theorem 2.8 (Lie's Third Fundamental Theorem)**.** *Let $\mathfrak{g}$ be an n-dimensional Lie algebra with basis vectors F_i, $1 \leq i \leq n$ such that $[F_j, F_k] = \frac{1}{2}C^i_{jk}F_i$. Then there is a unique local Lie group $N \subset \mathbb{R}^n$ such that $\mathbf{T}_0N$ and $\mathfrak{g}$ are isomorphic as Lie algebras.*

Note that the condition that $\mathfrak{g}$ is a Lie algebra is really just the condition that the bracket coefficients C^i_{jk} satisfy $C^i_{jk} = -C^i_{kj}$ (skew-symmetry) and $C^i_{pj}C^p_{kl} + C^i_{pk}C^p_{lj} + C^i_{pl}C^p_{jk} = 0$ (Jacobi identity). The local Lie group N is unique in the sense that any Lie group G with Lie algebra $\mathfrak{g}$ has a neighborhood of the identity that is isomorphic to N. A clear summary of the classical PDE proof of Lie's third fundamental theorem, including some of the corresponding EDS language, appears in [SW93, Chapter 8], and a version of the proof that inspires the corresponding theorem for Lie algebroids appears in [FC06].

Traditionally, Lie's theorem is presented as stated above, using brackets of vector fields, but the language of differential forms is more convenient for our purpose. Before proceeding to Cartan's generalization, let us bridge a conceptual gap and introduce some terminology by restating Lie's result in this language.

Theorem 2.9 (Lie's Third Fundamental Theorem, rephrased)**.** *Let C^i_{jk} be constants for $1 \leq i, j, k \leq n$. Consider the following structure equations:*

$$\mathrm{d}\alpha^i = -\frac{1}{2} C^i_{jk}\ \alpha^j \wedge \alpha^k \tag{2.11}$$

If $d^2 \equiv 0$ is satisfied on Equation (2.11)*, then there exists a solution manifold N^n with smooth co-framing (α^i) satisfying Equation* (2.11)*. Moreover, the solution (N, α) is locally unique up to structure-preserving diffeomorphism.*

The constants C^i_{jk} may be taken to be skew in the lower indices, since they always appear with a wedge product. The abused phrase "$d^2 \equiv 0$" is shorthand for saying that the constants C^i_{jk} are such that $0 \equiv \mathrm{d}(-\frac{1}{2}C^i_{jk}\alpha^j \wedge \alpha^k)$ modulo the relation in Equation (2.11). This condition is exactly the Jacobi identity. The equally abused phrase "locally unique up to structure-preserving diffeomorphism" means this: any two solutions (N, α) and $(\hat{N}, \hat{\alpha})$ have neighborhoods U and $\hat{U}$ of the identity that admit a diffeomorphism $\varphi : U \to \hat{U}$ such that $\varphi^*(\hat{\alpha}) = \alpha$. Of course, φ must be an isomorphism of local Lie groups, since N and $\hat{N}$ are both Lie groups.

Cartan's generalization deals with structure equations for which the C^i_{jk} are not constant but are instead functions $\mathbb{R}^s \to \mathbb{R}$.

Theorem 2.10 (Cartan's Structure Theorem)**.** *Let $V \subset \mathbb{R}^s$ be an open subset, and let C^i_{jk} and F^a_i be smooth functions on V for $1 \le i, j, k \le n$ and $1 \le a \le s$. Consider the following structure equations:*

$$\begin{aligned} \mathrm{d}\alpha^i &= -\frac{1}{2} C^i_{jk}(h^1, \ldots, h^s)\ \alpha^j \wedge \alpha^k, \\ \mathrm{d}h^a &= F^a_k(h^1, \ldots, h^s)\ \alpha^k. \end{aligned} \tag{2.12}$$

If $d^2 \equiv 0$ is satisfied on Equation (2.12)*, then for every $v \in V$ there exists a solution manifold N^n with smooth co-framing (α^i) and a smooth map $h : N \to V$ satisfying Equation* (2.12) *such that $v \in h(N)$. Moreover, for fixed v, the solution (N, α, h) is locally unique up to structure-preserving diffeomorphism.*

Again, the C^i_{jk} may as well be skew-symmetric, and the phrase "$d^2 \equiv 0$" means that differentiating the right-hand side of Equation (2.12) yields zero modulo the relations in Equation (2.12). The phrase "locally unique up to structure-preserving diffeomorphism [at v]" carries the fairly obvious meaning and is investigated further in Corollary 2.15 below.

This version of the structure theorem can be proven in the smooth category, but Cartan's most general version of the theorem uses Cartan–Kähler technology and apparently applies only in the real-analytic category. As pointed out in [FC06], the full relationship between Cartan's structure theorem and the integration theory of Lie algebroids merits further study.

The proof outlined below is decomposed into several lemmas, and the details that are provided require only smoothness.

First, some simple definitions. Consider only V and the vector fields $\{F_i = F^a_i \frac{\partial}{\partial x^a}\}_{i=1}^n$ where V has coordinates $x^1, \ldots, x^s$. These vector fields generate a singular distribution $\Delta = \operatorname{span}\{F_i\}$. At each point $v \in V$, let the *F-rank* at v be $r_F(v) = \dim \Delta_v$. If $r_F(v)$ were locally constant (that is, continuous) with value r, then the Frobenius theorem would imply the foliation of V by integral sub-manifolds of dimension r. However, in our case Δ may be singular. Of course, $r_F(v)$ is determined by the rank of the matrix $[F_1(v), \ldots, F_n(v)]$, which is given by the minor polynomials of the matrix F. Therefore the set where $r_F(v)$ is maximized is open and dense, and for all $k < \min\{n, s\}$, the set $\{x \ : \ r_F(v) \leq k\}$ is a closed subset of $\{v \ : \ r_F(v) \leq k{+}1\}$. In other words, $r : V \to \mathbb{N}$ is lower-semi-continuous.

An *F-curve* is defined to be a smooth curve $\gamma : [a, b] \to V$ such that $\gamma'(t) = c^i(t)F_i(\gamma(t))$ for some smooth functions c^i on $[a, b]$. The *F-leaf* containing x is defined as

$$\mathcal{O}_F(v) = \{\hat{v} \ : \ \exists \ F\text{-curve } \gamma, \gamma(a) = v, \gamma(b) = \hat{v}\}. \tag{2.13}$$

In the analytic category, these singular distributions were successfully proven to be integrable by singular foliations by Stefan and Sussmann [Sus73a] [Sus73b] [Ste74] [Ste80]. The next two lemmas rephrase the relevant details of their work.

Lemma 2.11 (Rank of leaves)**.** *If $\gamma : [a, b] \to V$ is an F-curve, then $r_F(\gamma(a)) = r_F(\gamma(t))$ for all $t \in [a, b]$. In particular, the F-rank is constant on each F-leaf.*

In the analytic category, this can be proven by writing a generic annihilator of Δ as a function on γ and showing that its Taylor series must vanish at $\gamma(a)$; thus it is demonstrated that $\Delta_{\gamma(b)}$ is isomorphic to $\Delta_{\gamma(a)}$ as vector spaces.

Lemma 2.12 (Singular Foliation)**.** *Each F-leaf is a connected, smooth manifold.*

Proof. Fix $v \in V$ and set $r = r_F(v)$. $\mathcal{O}_F(v)$ is connected by definition, and near v there are local coordinates $\psi_v : \mathbb{R}^r \to \mathcal{O}_F(v)$ given by flow-box coordinates on a basis $\{F_1, \ldots, F_r\}$ of Δ_v. All that remains is to show that these maps ψ_v are smooth on overlaps, as accomplished in [Ste74]. $\square$

In the modern language of algebroids and groupoids, consider a Lie algebroid $A = V \times \mathbb{R}^n$ with anchor map to $\mathbf{T}V$ given by the vector fields $\{F_i\}$. Then the singular distribution Δ is the image of the anchor map. Generally, this Lie algebroid is neither regular nor transitive. With some technical care regarding composition of arrows and concatenation of F-curves, an F-curve in V is essentially an arrow for a Lie groupoid $\mathcal{G}$ over V that integrates A. These two Lemmas amount to the statement that, given a fixed source v, the targets of arrows (the endpoints of F-curves) form the set $\mathcal{O}_F(v)$, which is a submanifold of V [Mac05, Theorem 1.5.11] and that the Lie groupoid $\mathcal{G}$ restricts to each leaf $\mathcal{O}_F(v)$ as a locally trivial (that is, transitive) Lie groupoid [Mac05, Theorem 1.5.12].

Whichever perspective is used, a "singular Frobenius theorem" is in place, and solutions to the structure equations can be studied.

Lemma 2.13. *If (N, α, h) is a connected solution to the structure system (V, C, F), then $h : N \to h(N)$ is a submersion into a single F-leaf.*

Proof. For any $p \in N$ the image of $\mathrm{d}h(p)$ is given by the span of $\{F_i(h(p))\}$, which is exactly the tangent-space of the F-leaf containing $h(p)$. The image of connected N must remain in a single leaf, as a path in $\gamma : [0, 1] \to N$ produces a path $h \circ \gamma$ in N with $(h \circ \gamma) = \mathrm{d}h(\gamma'(t))$, so γ is a path whose velocity is may be written as $c^i(t)F_i(\gamma(t))$. $\square$

The key to constructing solutions is the symmetry algebra of a point in the leaf, which we now construct: Fix $v \in V$ and consider the map $\lambda_v : \mathbb{R}^n \to \mathbf{T}_v\mathcal{O}_F(v)$ given by $\lambda_v : e_i \mapsto F_i$. Define a Lie bracket on $\mathbb{R}^n$ by $[e_j, e_k] = \frac{1}{2}C^i_{jk}(v)e_i$. The *symmetry algebra at v* is $\mathfrak{h}_v = \ker \lambda_v$, which is a Lie algebra with this Lie bracket.

Lemma 2.14 (Existence of Solutions)**.** *Suppose (N, α, h) is a connected and simply connected solution of (V, C, F) with $p \in N$ and $v = h(p)$. There is a neighborhood $U \ni p$ that is diffeomorphic to $L \times H$ where $L \subset \mathcal{O}_F(v)$ is a contractible neighborhood of v and H is an open Lie group with Lie algebra $\mathfrak{h}_v$.*

Proof. Fix $p \in N$ with $r = r_F(v)$. Some $r \times r$ sub-matrix of (F^a_i) has maximum rank at p. This is an open condition, so in a neighborhood U of p, the same $r \times r$ sub-matrix of (F^a_i) maintains maximum rank. By reordering the coordinates on V, we may assume $\mathrm{d}h^1 \wedge \cdots \wedge \mathrm{d}h^r \neq 0$. Additionally, we may assume $\mathrm{d}h^1 \wedge \cdots \wedge \mathrm{d}h^r \wedge \mathrm{d}h^i = 0$ for all $i > r$, as otherwise we would have $r_F(v) > r$.

Let K be the distribution on U that is annihilated by h_*. Of course, $0 = h_*(w) = \mathrm{d}h(w)$, so $w \in K$ if and only if $0 = \mathrm{d}h^1(w) = \cdots = \mathrm{d}h^r(w)$. The EDS algebraically generated by $\{\mathrm{d}h^1, \ldots, \mathrm{d}h^r\}$ satisfies the conditions of the Frobenius theorem, so the distribution K is integrable, meaning U (or a smaller open subset) is smoothly foliated by integral submanifolds of dimension $n - r$. In particular there are local coordinates $(u^1, \ldots, u^n)$ on U centered at p such that the integral manifolds of the Frobenius system are given by $u^1 = C_1$, $u^2 = C_2$, $\ldots$, $u^r = C_r$ for r constants $\{C_i\}$. Let H denote the leaf through p, so $\mathbf{T}_pH = K_p$. Let Σ denote a submanifold of dimension r transverse to the leaf H. U is the total-space of a bundle with fiber H over a base manifold Σ.

Let $\{E_i\}$ denote the frame dual to the co-frame $\{\alpha^i\}$ on U. Define a skew bilinear form $[\cdot,\cdot]$ on $\Gamma(\mathbf{T}U)$ by $[E_j, E_k] = \frac{1}{2}(C^i_{j,k} \circ h)E_i$. Notice that $C^i_{jk} \circ h$ is constant on H; therefore, K is a Lie algebra with bracket $[\cdot,\cdot]$. In fact, $K = \mathfrak{h}_v$, so $\mathbf{T}_{(u^i)}H = \mathfrak{h}_v$ for all (u^i). By Theorem 2.8, H a local Lie group integrating $\mathfrak{h}_v$.

Finally, Σ is diffeomorphic to an open subset L of the leaf $\mathcal{O}_F(v)$, since Σ is transverse to H, so $(h|_\Sigma)_*$ is an isomorphism. $\square$

In this proof, existence is shown by constructing a bundle $L \times H$ on which the structure equations hold and where H is a local Lie group with Lie algebra $\mathfrak{h}_v$. This construction provides a local uniqueness statement.

Corollary 2.15 (Local uniqueness of solutions)**.** *Let (N, α, h) and $(\hat{N}, \hat{\alpha}, \hat{h})$ be solutions of the structure system (V, C, F) with $h(N) \cap \hat{h}(\hat{N}) \neq \emptyset$. For any p, $\hat{p}$ such that*

$h(p) = v = \hat{h}(\hat{p})$, *there exist neighborhoods* $U \ni p$ *and* $\hat{U} \ni \hat{p}$ *and a diffeomorphism* $\varphi : U \to \hat{U}$ *such that* $\varphi^*(\hat{\alpha}) = \alpha$ *and* $\hat{h} \circ \varphi = h$.

In the situation of Corollary 2.15, we say that the solutions (N, α, h) and $(\hat{N}, \hat{\alpha}, \hat{h})$ *represent* v. For fixed v, solutions that represent v form an equivalence class.

These have been local statements, and there may be topological obstructions to the existence of N with $h(N) = \mathcal{O}_F(v)$. To avoid concerning ourselves with these obstructions, it is useful to define a finite-sequence version of equivalence.

Definition 2.16 (Leaf equivalence)**.** *Two solutions* (N_0, α_0, h_0) *and* (N_k, α_k, h_k) *to* (V, C, F) *are said to be* leaf-equivalent *if there exist connected solutions* (N_i, α_i, h_i) *with* $h(N_i) \cap h(N_{i-1}) \neq \emptyset$ *for* $i = 1, \ldots, k$.

In fact, the proof of Lemma 2.14 also shows that $\mathfrak{h}_v$ is locally constant within the F-leaves of V. In particular, since each leaf is connected, $\mathfrak{h}_v$ depends only on $\mathcal{O}_F(v)$. Therefore, solutions (N_0, α_0, h_0) and (N_k, α_k, h_k) that are leaf-equivalent on $\mathcal{O}_F(v)$ must have neighborhoods diffeomorphic to the bundle $L \times H$ where H is the local Lie group for the symmetry algebra $\mathfrak{h}_v$; however, the diffeomorphism cannot preserve α_i and h_i unless the solutions both represent the same $v \in \mathcal{O}_F(v)$.

3

GL(2)-Structures and their Representations

In this chapter, we specify notation for representations of $GL(2,\mathbb{R})$ and for $GL(2,\mathbb{R})$-structures. This notation is used extensively throughout the following material. With the exception of Chapter 7, all groups that appear in this dissertation are real, so $GL(2,\mathbb{R})$ is henceforth denoted $GL(2)$ without confusion.

3.1 Binary Polynomials

Let $\mathcal{V}_n \subset \mathbb{R}[x,y]$ denote the vector space of degree n homogeneous polynomials in x and y with real coefficients. We identify $\mathcal{V}_n$ with $\mathbb{R}^{n+1}$ using the terms from the binomial theorem to produce a basis; for example, $\mathcal{V}_2 \to \mathbb{R}^3$ by

$$v_{-2}\ x^2 + v_0\ 2xy + v_2\ y^2 \mapsto (v_{-2}, v_0, v_2) \in \mathbb{R}^3. \tag{3.1}$$

Recall that $\mathcal{V}_n$ is the unique irreducible representation of $\mathfrak{sl}(2)$ of dimension $n+1$, and its action is generated by

$$\mathbf{X} = y\frac{\partial}{\partial x},\ \mathbf{Y} = -x\frac{\partial}{\partial y},\ \text{and}\ \mathbf{H} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{3.2}$$

Furthermore, the Clebsch–Gordon [Hum72] pairings $\langle\cdot,\cdot\rangle_p : \mathcal{V}_m \otimes \mathcal{V}_n \to \mathcal{V}_{m+n-2p}$ are given by the formula

$$\langle u, v\rangle_p = \frac{1}{p!}\sum_{k=0}^{p}(-1)^k \binom{p}{k} \frac{\partial^p u}{\partial x^{p-k}\partial y^k} \cdot \frac{\partial^p v}{\partial x^k \partial y^{p-k}} \tag{3.3}$$

and describe the decomposition of the tensor product into irreducible components

$$\mathcal{V}_m \otimes \mathcal{V}_n = \mathcal{V}_{|m-n|} \oplus \mathcal{V}_{|m-n|+2} \oplus \cdots \oplus \mathcal{V}_{m+n-2} \oplus \mathcal{V}_{m+n}. \tag{3.4}$$

This pairing has some important properties. Notice that $\langle u, v\rangle_p = (-1)^p \langle v, u\rangle_p$ and that the pairing is nontrivial for $0 \leq p \leq \min\{m, n\}$. Hence, the tensor decomposition can be further refined in terms of the symmetric and alternating tensors:

$$\mathcal{V}_n \circ \mathcal{V}_n = \mathcal{V}_{2n} \oplus \mathcal{V}_{2n-4} \oplus \cdots \oplus \mathcal{V}_{0 \text{ or } 2}, \tag{3.5}$$

$$\mathcal{V}_n \wedge \mathcal{V}_n = \mathcal{V}_{2n-2} \oplus \mathcal{V}_{2n-6} \oplus \cdots \oplus \mathcal{V}_{2 \text{ or } 0}. \tag{3.6}$$

Notice too that $\langle\cdot,\cdot\rangle_n : \mathcal{V}_n \otimes \mathcal{V}_n \to \mathcal{V}_0 = \mathbb{R}$ is a non-degenerate symmetric- or skew-bilinear form. Hence, for fixed $u \in \mathcal{V}_n$ the map $\langle u, \cdot\rangle_n : \mathcal{V}_n \to \mathcal{V}_0 = \mathbb{R}^1$ provides is a natural identification, $\mathcal{V}_n = \mathcal{V}_n^*$, and we never distinguish between dual spaces when considering representations.

For any derivation over $\mathbb{R}[x, y]$, a Leibniz rule over the pairing holds. Because $SL(2)$ is infinitesimally generated by $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{H}$, this means that the pairings are $SL(2)$-equivariant. That is, $\alpha(\langle u, v\rangle_p) = \langle \alpha(u), v\rangle_p + \langle u, \alpha(v)\rangle_p$ for any $\alpha \in \mathfrak{sl}(2)$ implies $a \cdot \langle u, v\rangle_p = \langle a \cdot u, a \cdot v \cdot a\rangle_p$ for any $a \in SL(2)$.

The pairing can be generalized to binary-polynomial-valued alternating forms over a manifold. If $u \in \Gamma(\wedge^r \mathbf{T}^* M \otimes \mathcal{V}_m)$ and $v \in \Gamma(\wedge^s \mathbf{T}^* M \otimes \mathcal{V}_n)$, then extend the definition by using the wedge-product:

$$\langle u, v\rangle_p = \frac{1}{p!}\sum_{k=0}^{p}(-1)^k \binom{p}{k} \frac{\partial^p u}{\partial x^{p-k}\partial y^k} \wedge \frac{\partial^p v}{\partial x^k \partial y^{p-k}}. \tag{3.7}$$

In this generalization, the symmetry of the pairing is further altered by the degree of the forms: $\langle u, v\rangle_p = (-1)^{rs+p} \langle v, u\rangle_p$.

Let $\mathbf{I}$ denote the identity map on $\mathcal{V}_n$. Then $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{R}\mathbf{I} = \mathcal{V}_2 \oplus \mathcal{V}_0$ is the Lie algebra of the $GL(2)$ generated by the actions of $SL(2)$ and scaling in $\mathcal{V}_n$. The pairing is not $GL(2)$-equivariant, but the scaling action is computed easily where needed, as in Equation (4.17). The geometric objects we encounter are projectively defined, so the variance in scaling is generally of little concern. Notably, if λ is a $\mathbb{R}\mathbf{I}$-valued 1-form, then $\lambda \wedge \omega$ may be written as the trivial pairing $\langle \lambda, \omega\rangle_0 = (-1) \langle \omega, \lambda\rangle_0 \in \Gamma(\wedge^2 \mathbf{T}^* M \otimes \mathcal{V}_n)$ for any $\mathcal{V}_n$-valued 1-form ω.

The vector space $\mathcal{V}_n$ admits two particularly interesting $GL(2)$-invariant subsets. Let $\mathcal{C} \subset \mathcal{V}_n$ denote the set of perfect nth degree polynomials, and let $\mathcal{Q} \subset \mathcal{V}_n$ denote the set of polynomials where a root is repeated $k-1$ times. That is,

$$\begin{aligned} \mathcal{C} &= \{(ax + by)^k\}, \text{ and} \\ \mathcal{Q} &= \{(ax + by)^{k-1}(Ax + By)\}. \end{aligned} \tag{3.8}$$

It is not hard to see that $\mathcal{C}$ is a rational normal cone of dimension 2; $\mathcal{Q}$ is the tangent developable of $\mathcal{C}$, and $\mathcal{C}$ is the singular set of $\mathcal{Q}$. Each of 0, $\mathcal{C} \setminus 0$ and $\mathcal{Q} \setminus \mathcal{C}$ is a $GL(2)$-orbit in $\mathcal{V}_k$, and $\mathcal{C} \setminus 0$ is the unique orbit of dimension 2. In Chapter 7 the $GL(2)$-orbits in $\mathcal{V}_k$ are considered for the particularly interesting case of $k = 8$.

Before we begin the discussion of bundles and co-frames, this is a good time to mention an indexing convention used throughout this dissertation that may cause some confusion for the reader. Some objects are labeled with many sub- and superscripts, for example $R^{2,6}_4$. The indices denote specific irreducible components of R given by the $\mathfrak{gl}(2)$ representation and the Clebsch–Gordon decomposition. In this case R takes values in $(\mathcal{V}_2 \oplus \mathcal{V}_0) \otimes (\mathcal{V}^*_2 \oplus \mathcal{V}^*_6)$, and $R^{2,6}_4$ denotes the irreducible component of R occurring in the copy of $\mathcal{V}_4$ obtained from the Clebsch–Gordon decomposition of the product $\mathcal{V}_2 \otimes \mathcal{V}^*_6$. Another subscript appears when writing the full polynomial, and this subscript varies symmetrically across $\{-n, -n{+}2, \ldots, n{-}2, n\}$ to indicate the vector components in the irreducible representation of weight n:

$$R_4^{2,6} = R_{4,-4}^{2,6}\ x^4 + R_{4,-2}^{2,6}\ 4x^3y + R_{4,0}^{2,6}\ 6x^2y^2 + R_{4,2}^{2,6}\ 4xy^3 + R_{4,4}^{2,6}\ y^4. \tag{3.9}$$

In other situations, the super-script may vary in length depending upon how much information is necessary to uniquely determine the irreducible component using the Clebsch–Gordon formula.

3.2 GL(2)-Structures

Consider a real manifold M of dimension $n{+}1$. Let $\mathcal{F}$ denote the $\mathcal{V}_n$-valued co-frame bundle of M. That is, $\mathcal{F}$ is the bundle whose fibers are comprised of isomorphisms $u_p : \mathbf{T}_pM \to \mathcal{V}_n$. For $g \in GL(\mathcal{V}_n)$, the right action $u_p \cdot g = g^{-1} \circ u_p$ acts smoothly, simply and transitively on $\mathcal{F}_p$, so $\mathcal{F}$ is a principal right $GL(\mathcal{V}_n)$ bundle.

Definition 3.1 (GL(2)-structure). *A* $GL(2)$-structure *on an* $(n{+}1)$*-dimensional manifold* M *is a sub-bundle* $B \subset \mathcal{F}$ *whose fiber is infinitesimally spanned by the actions of* $\mathbf{X}$*,* $\mathbf{Y}$*,* $\mathbf{H}$*, and* $\mathbf{I}$ *on* $\mathcal{V}_n$*.* B *is a right-principal* $GL(2)$*-bundle over* M*. When the dimension of* M *is* $n{+}1$*, the* $GL(2)$*-structure is said to have* degree n*.*

The geometric content of a $GL(2)$-structure is provided by the following standard theorem.

Theorem 3.2. *A choice of* $GL(2)$*-structure on* M^{n+1} *is equivalent to a choice of a smooth field of rational normal cones,* $\mathbf{C} \subset \mathbf{T}M$*.*

Proof. Suppose $B \to M$ is a $GL(2)$-structure. Since B is a reduction of $\mathcal{F}(M)$, a point $b \in B_p$ is the $GL(2)$ orbit of an isomorphism $u_p : \mathbf{T}_pM \to \mathcal{V}_n$. The $GL(2)$ action on $\mathcal{V}_n$ yields a unique closed 2-dimensional orbit, a rational normal cone. Let $\mathbf{C}_p = u_p^{-1}(\mathcal{C})$. Since u is only defined up to a $GL(2)$ action and the symmetry group of $\mathcal{C}$ is the same action of $GL(2)$, the cone $\mathcal{C}_p$ is well-defined. $\mathbf{C}$ is therefore a smooth distribution of rational normal cones over M.

Conversely, suppose M is equipped with a distribution $\mathbf{C}$ of rational normal cones. By the definition of "rational normal cone," there exists at each point $p \in M$ an isomorphism $u_p : \mathbf{T}_p M \to \mathcal{V}_n$ that has the property $u_p(\mathbf{C}_p) = \mathcal{C}$. From Lemma 1.4, the symmetry group of $\mathcal{C}$ is $PGL(2)$, so any $g^{-1} \circ u_p \in u_p \cdot GL(2)$ also has this property. Hence, $\mathbf{C}$ specifies a $GL(2)$ reduction of $\mathcal{F}$. $\square$

Note that our notion of $GL(2)$-structures only includes the particular embedding of $GL(2)$ into $GL(\mathcal{V}_n)$ that is given as the symmetry group of the rational normal cone. One could consider a broader class of structures defined by various embeddings of $GL(2)$ into $GL(\mathbb{R}^{n+1})$ or a broader class of structures defined by the symmetry group of a different family of cones. Of course, cones over curves of degree $d < n$ are degenerate and fail to be interesting as they correspond to "flat" embeddings of $GL(2)$ into a subspace of $GL(\mathbb{R}^{d+1}) \subset GL(\mathbb{R}^{n+1})$. On the other hand, congruence classes of cones over non-degenerate curves of degree $d > n$ may be extremely complicated. The rational normal cone is both interesting and tractable, so for our purposes "$GL(2)$-structure" always means a structure given these actions of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{H}$, and $\mathbf{I}$ on $\mathcal{V}_n$ preserving $\mathcal{C}$.

The notion of *k-integrability* for a $GL(2)$-structure is introduced in Chapter 1. The goal of this dissertation is to understand and classify local k-integrable $GL(2)$-structures. The only interesting values of k are 2 and 3. If $k = 1$, then the local condition is easily satisfied, depending on one function of one variable. If $k = n$, then the infinitesimal condition is open. If $k \geq 4$, then not all k-secant subspaces are in the same $GL(2)$-orbit, so the infinitesimal problem alone is difficult. The $GL(2)$ orbits on $Gr_k(\mathcal{V}_n)$ are relatively new in the literature [CM09]. The dimension of M matters profoundly in the theory of $GL(2)$-structures, particularly regarding 2- and

3-integrability. The case $n = 2$ is trivial. The case $n = 3$ was thoroughly studied by Bryant in [Bry91]. Chapters 4 through 9 are concerned with 5-dimensional manifolds in order to examine the situation discovered by [FHK07], so the degree of the co-frame polynomials is $n = 4$ in those chapters. In Chapters 10 through 12, all $n \geq 5$ are considered, but firm results are only computed for $5 \leq n \leq 20$; however, one expects no surprises for $n \geq 21$, and extending the computations to that range should be merely a matter of picking apart some rather tedious combinatorics. Hence, this dissertation (almost) completes the theory of k-integrable $GL(2)$-structures.

In [Bry91], a related condition is also studied. Consider surfaces $\Sigma \subset M$ where $\mathbf{T}_p\Sigma$ is contained in $\mathcal{Q} \subset \mathcal{V}_n \simeq \mathbf{T}_pM$, so $\mathbf{T}_p\Sigma$ is tangent to $\mathcal{C}$ (instead of 2-secant to $\mathcal{C}$). This is essentially the condition that arises for 4th order ($n = 3$) ODE geometry in [Bry91] and for 5th order ($n = 4$) ODE geometry in [GN06], [Nur07], [BN07], [GN07], and [GN09]. There is no direct translation between k-integrability and this condition; indeed, it appears that the two conditions are mutually exclusive for a given $GL(2)$-structure, as the essential torsion invariants take values in different irreducible representations of $GL(2)$. However, the systems are eerily similar and their relationship should be examined.

4

Equivalence in Degree Four

In this chapter, we prove the existence of a canonical co-framing for a $GL(2)$-structure of degree four. This co-framing is later used in Chapters 5 and 6 to study the differential ideal that describes the existence of bi-secant surfaces and tri-secant 3-folds.

4.1 The Tautological Form

Consider a manifold M of dimension 5 and its $\mathcal{V}_4$-valued co-frame bundle $\mathcal{F}$ whose fibers are comprised of isomorphisms $u_p : \mathbf{T}_p M \to \mathcal{V}_4$. $\mathcal{F}$ is a principal right $GL(5)$ bundle. We want to study the equivalence of $GL(2)$-structures over M. Let $B \subset \mathcal{F}$ denote a $GL(2)$-structure generated by $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{H}$, and $\mathbf{I}$ as in Chapter 3. We follow the method described in Chapter 2.

The tautological form of $\mathcal{F}$ is the unique and global 1-form naturally defined by $\omega_u = u \circ \pi : \mathbf{T}_u\mathcal{F} \to \mathcal{V}_n$. The tautological form is semi-basic (that is, it annihilates π-vertical vectors) and it pulls back to any sub-bundle as a unique, global and semi-basic 1-form. So, let ω also denote the tautological form of B, which may written in vector or polynomial form using the earlier identification of $\mathbb{R}^{n+1}$ with $\mathcal{V}_n$ as

$$\omega = \begin{pmatrix} \omega^{-4} \\ \omega^{-2} \\ \omega^{0} \\ \omega^{2} \\ \omega^{4} \end{pmatrix} = \omega^{-4}\ x^4 + \omega^{-2}\ 4x^3y + \omega^0\ 6x^2y^2 + \omega^2\ 4xy^3 + \omega^4\ y^4. \qquad (4.1)$$

Since $\omega \in \Gamma(\mathbf{T}^*B \otimes \mathcal{V}_n)$, we follow the convention that ω is a column-vector with indices raised. Notice that $\mathrm{d}\omega \in \Gamma(\wedge^2[\mathbf{T}^*B] \otimes \mathcal{V}_4)$ and $\omega \wedge \omega \in \Gamma(\wedge^2[\mathbf{T}^*B \otimes \mathcal{V}_4])$. The latter can be rewritten via the Clebsch–Gordon decomposition as

$$\langle \omega, \omega \rangle_3 + \langle \omega, \omega \rangle_1 \in \Gamma(\wedge^2(\mathbf{T}^*B) \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6)). \qquad (4.2)$$

4.2 Connection

A connection for this bundle is a $\mathfrak{gl}(2)$-valued 1-form on B, $\theta \in \Gamma(\mathbf{T}^*B \otimes \mathfrak{gl}(2))$. Using the natural decomposition of $\mathfrak{gl}(2)$ into $\mathfrak{sl}(2) \oplus \mathbb{R}$, θ may be re-written as $\varphi + \lambda \in \Gamma(\mathbf{T}^*B \otimes (\mathcal{V}_2 \oplus \mathcal{V}_0))$. Using this decomposition, $\theta \wedge \omega \in \Gamma(\wedge^2(\mathbf{T}^*B) \otimes \mathcal{V}_4)$ is computed using the Clebsch–Gordon pairing as $\langle \varphi, \omega \rangle_1 + \langle \lambda, \omega \rangle_0 \in \Gamma(\wedge^2(\mathbf{T}^*B) \otimes \mathcal{V}_4)$. In particular, writing $\varphi = \varphi_{-2}\ x^2 + \varphi_0\ 2xy + \varphi_2\ y^2 \in \mathcal{V}_2$ and $\lambda \in \mathcal{V}_0$ provides the first structure equation

$$\mathrm{d}\omega = -\langle \varphi, \omega \rangle_1 - \langle \lambda, \omega \rangle_0 + T(\omega \wedge \omega) \qquad (4.3)$$

where $T \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4^*)$. Equation (4.3) may also be written in matrix form as $\mathrm{d}\omega^i = -\theta^i_j \wedge \omega^j + T(\omega \wedge \omega)$. In coordinates, $-\theta \wedge \omega$ is

$$-\begin{pmatrix} 8\varphi_0 - \lambda & -8\varphi_{-2} & 0 & 0 & 0 \\ 2\varphi_2 & 4\varphi_0 - \lambda & -6\varphi_{-2} & 0 & 0 \\ 0 & 4\varphi_2 & -\lambda & -4\varphi_{-2} & 0 \\ 0 & 0 & 6\varphi_2 & -4\varphi_0 - \lambda & -2\varphi_{-2} \\ 0 & 0 & 0 & 8\varphi_2 & -8\varphi_0 - \lambda \end{pmatrix} \wedge \begin{pmatrix} \omega^{-4} \\ \omega^{-2} \\ \omega^{0} \\ \omega^{2} \\ \omega^{4} \end{pmatrix}. \qquad (4.4)$$

So, the matrix representation of θ is $(2\varphi_{-2}\mathbf{X} + 2\varphi_0\mathbf{H} + 2\varphi_2\mathbf{Y} + \lambda\mathbf{I})$, which takes values in $\mathfrak{gl}(2) \subset \mathfrak{gl}(\mathcal{V}_4)$.

4.3 Normalization of Torsion

The remainder of this chapter establishes Theorem 4.1, which provides a unique connection and global canonical co-framing for B.

Changes of connection are of the form $\hat{\varphi} = \varphi + P(\omega)$ and $\hat{\lambda} = \lambda + Q(\omega)$ where $P \in \mathcal{V}_2 \otimes \mathcal{V}_4^* = \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6$ and $Q \in \mathcal{V}_0 \otimes \mathcal{V}_4^* = \mathcal{V}_4$. A canonical connection is obtained by analyzing the exact sequence

$$0 \to \mathfrak{gl}(2)^{(1)} \to (\mathcal{V}_2 \oplus \mathcal{V}_0) \otimes \mathcal{V}_4 \xrightarrow{\delta} \mathcal{V}_4 \otimes (\wedge^2 \mathcal{V}_4) \to H^{0,2}(\mathfrak{gl}(2)) \to 0. \tag{4.5}$$

We must compute the images, δP and δQ, to find $H^{0,2}$, and we are aided by the Clebsch–Gordon formula. It is now time to start decomposing and labeling our various objects. First, the torsion:

$$T \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4^*) = \mathcal{V}_4 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6) = (\mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6) \oplus (\mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}). \tag{4.6}$$

Hence, using the notation introduced in Chapter 3, we write $T = (T_2^2 + T_4^2 + T_6^2) + (T_2^6 + T_4^6 + T_6^6 + T_8^6 + T_{10}^6)$ where $T_2^6 = T_{2,-2}^6\, x^2 + T_{2,0}^6\, 2xy + T_{2,2}^6\, y^2 \in \mathcal{V}_2 \subset \mathcal{V}_4 \otimes \mathcal{V}_6$, and so on. Then we may fully decompose the torsion as

$$\begin{aligned} T(\omega,\omega) = {} & \left\langle T_2^2, \langle w,w\rangle_3 \right\rangle_0 + \left\langle T_4^2, \langle w,w\rangle_3 \right\rangle_1 + \left\langle T_6^2, \langle w,w\rangle_3 \right\rangle_2 \\ & + \left\langle T_2^6, \langle w,w\rangle_1 \right\rangle_2 + \left\langle T_4^6, \langle w,w\rangle_1 \right\rangle_3 + \left\langle T_6^6, \langle w,w\rangle_1 \right\rangle_4 \\ & + \left\langle T_8^6, \langle w,w\rangle_1 \right\rangle_5 + \left\langle T_{10}^6, \langle w,w\rangle_1 \right\rangle_6 . \end{aligned} \tag{4.7}$$

Now, consider the change-of-connection, $P \in \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6$, where $\mathcal{V}_2 \ni P(\omega) = \langle P_2, \omega\rangle_2 + \langle P_4, \omega\rangle_3 + \langle P_6, \omega\rangle_4$. Let $\delta P \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4)$ have components $\delta P = \delta P_2^2 + \delta P_4^2 + \delta P_6^2 + \delta P_2^6 + \delta P_4^6 + \delta P_6^6 + \delta P_8^6 + \delta P_{10}^6$, similar to the decomposition of T. As δ is a linear map, each of δP_k^2 and δP_k^6 is a linear combination of the various P_j. Moreover, δP_k^2 and δP_k^6 can only be in the image of P_j for $j = k$, since these are irreducible representations and the action of δ must be $SL(2)$-equivariant. That is,

δ must preserve the weights of the representations. In particular there must exist constants a_2, a_4, a_6, b_2, b_4, and b_6 such that

$$\begin{aligned} 0 &= \langle P(\omega), \omega \rangle_1 - \delta P(\omega, \omega) \\ &= \langle \langle P_2, \omega \rangle_2, \omega \rangle_1 + \langle \langle P_4, \omega \rangle_3, \omega \rangle_1 + \langle \langle P_6, \omega \rangle_4, \omega \rangle_1 \\ &- \langle a_2 P_2, \langle w, w \rangle_3 \rangle_0 - \langle a_4 P_4, \langle w, w \rangle_3 \rangle_1 - \langle a_6 P_6, \langle w, w \rangle_3 \rangle_2 \\ &- \langle b_2 P_2, \langle w, w \rangle_1 \rangle_2 - \langle b_4 P_4, \langle w, w \rangle_1 \rangle_3 - \langle b_6 P_6, \langle w, w \rangle_1 \rangle_4 . \end{aligned} \tag{4.8}$$

Carrying out this computation shows that

$$a_2 = \frac{3}{10},\ b_2 = \frac{1}{5},\ a_4 = \frac{1}{2},\ b_4 = 0,\ a_6 = -\frac{1}{5},\ b_6 = -\frac{1}{20}. \tag{4.9}$$

Similarly (but more easily) we see that $Q \in \mathcal{V}_4$, where $\mathcal{V}_0 \ni Q(\omega) = \langle Q_4, \omega \rangle_4$. Let $\delta Q \in \mathcal{V}_4 \otimes \wedge^2(\mathcal{V}_4)$ have components $\delta Q = \delta Q^2_2 + \delta Q^2_4 + \delta Q^2_6 + \delta Q^6_2 + \delta Q^6_4 + \delta Q^6_6 + \delta Q^6_8 + \delta Q^6_{10}$, but again the image must have the same weight as the domain. In particular there must exist constants c_4, and d_4 such that

$$\begin{aligned} 0 &= \langle Q(\omega), \omega \rangle_0 - \delta Q(\omega, \omega) \\ &= \langle \langle Q_4, \omega \rangle_4, \omega \rangle_0 - \langle c_4 Q_4, \langle w, w \rangle_3 \rangle_1 - \langle d_4 Q_4, \langle w, w \rangle_1 \rangle_3 \end{aligned} \tag{4.10}$$

Carrying out this computation shows that

$$c_4 = -\frac{1}{40},\ d_4 = -\frac{1}{160}. \tag{4.11}$$

These computations allow us to normalize the torsion.

Theorem 4.1. *$\mathfrak{gl}(2)^{(1)} = \mathfrak{sl}(2)^{(1)} = 0$ and $H^{0,2}(\mathfrak{sl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$, while $H^{0,2}(\mathfrak{gl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$. In particular, any $GL(2)$-structure B over M^5 admits four distinct connections such that the torsion T is of the form*

$$T = T_2 + T_6 + T_8 + T_{10} \in \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10} \subset \mathcal{V}_4 \otimes (\mathcal{V}_2 \oplus \mathcal{V}^*_6). \tag{4.12}$$

Proof. Let (φ, λ) be an arbitrary connection with torsion T, and let $\hat{\varphi} = \varphi + P(\omega)$, $\hat{\lambda} = \lambda + Q(\lambda)$ have torsion $\hat{T}$. Then

$$\begin{aligned} \hat{T}(\omega, \omega) &= \mathrm{d}\omega + \langle \hat{\varphi}, \omega \rangle_1 + \left\langle \hat{\lambda}, \omega \right\rangle_0 \\ &= \mathrm{d}\omega + \langle \varphi, \omega \rangle_1 + \langle P(\omega), \omega \rangle_1 - \langle Q(\omega), \omega \rangle_0 \\ &= (T + \delta P + \delta Q)(\omega, \omega). \end{aligned} \tag{4.13}$$

Using Equation (4.9) and Equation (4.11), the absorption of torsion is dictated by the solvability of the equations

$$\begin{aligned} \hat{T}^2_2 &= T^2_2 + \frac{3}{10} P_2, \\ \hat{T}^2_4 &= T^2_4 + \frac{1}{2} P_4 - \frac{1}{40} Q_4, \\ \hat{T}^2_6 &= T^2_6 - \frac{1}{5} P_6, \\ \hat{T}^6_2 &= T^6_6 + \frac{1}{5} P_2, \\ \hat{T}^6_4 &= T^6_4 - \frac{1}{160} Q_4, \\ \hat{T}^6_6 &= T^6_6 - \frac{1}{20} P_6, \\ \hat{T}^6_8 &= T^6_8, \\ \hat{T}^6_{10} &= T^6_{10}. \end{aligned} \tag{4.14}$$

Generally, we may choose P_2 to force exactly one of $\hat{T}^6_2$ or $\hat{T}^2_2$ to vanish. Similarly, we may choose P_6 to force exactly one of $\hat{T}^6_6$ or $\hat{T}^2_6$ to vanish. Unique Q_4 and P_4 eliminate $\hat{T}^6_4$ and $\hat{T}^2_4$. All other components of $\hat{T}$ are fixed. $\square$

Note that in fact there are *four* canonical connections on B with torsion in $H^{0,2}(\mathfrak{gl}(2))$, based on the choice of which representations of T to absorb in weight 2 and 6. For concreteness in computation, we make a consistent choice.

Corollary 4.2. *There is a unique choice of Q_4, P_2, P_4, and P_6 such that $T_4^6 = T_2^2 = T_4^2 = T_6^2 = 0$.*

Proof. Choose P_2 to set $\hat{T}_2^2 = 0$, and choose P_6 to set $\hat{T}_6^2 = 0$. □

This solves the equivalence problem and establishes a global and canonical (if not entirely unique) co-framing for B. Henceforth, the connection (φ, λ) is assumed to be this unique connection. The first structure equation, represented as Equation (4.4) or Equation (4.3), still holds, where we understand that $T = T_2 \oplus T_6 \oplus T_8 \oplus T_{10}$ as above, and we write the components of T_k as

$$T_k = (T_{k,-k}, T_{k,-k+2}, \ldots, T_{k,k-2}, T_{k,k}) = \sum_{j=0}^{k} T_{k,2j-k} \binom{k}{j} x^{k-j} y^j. \tag{4.15}$$

The Clebsch–Gordon pairing is not $GL(2)$-equivariant; to see how a scaling action affects the torsion, fix $\rho \in \mathbb{R}^\times$, and let $R_g(u) = u \cdot g = g^{-1}u$ denote the right action of $g \in GL(2)$ on $u \in B$. Then $R^*_{\rho I}(\omega) = \rho^{-1}\omega$, and the first structure equation transforms as

$$R^*_{\rho I}(\mathrm{d}\omega) = \mathrm{d}(R^*_{\rho I}\omega) = \mathrm{d}(\rho^{-1}\omega) = \rho^{-1}(\mathrm{d}\omega). \tag{4.16}$$

This allows us to derive the action of ρI in the torsion. For example,

$$\rho^{-1} \langle T_8, \langle \omega, \omega \rangle_1 \rangle_5 = R^*_{\rho I} \left(\langle T_8, \langle \omega, \omega \rangle_1 \rangle_5 \right) = \left\langle R^*_{\rho I}(T_8), \left\langle \rho^{-1}\omega, \rho^{-1}\omega \right\rangle_1 \right\rangle_5. \tag{4.17}$$

In particular, $R^*_{\rho I}(T_k) = \rho T_k$ for any irreducible component T_k of T of degree k.

4.4 Curvature and the Bianchi Identity

Second-order information can be obtained by differentiating the first structure equation and studying the Bianchi identity, $\nabla(\theta) \wedge \omega = \nabla(T(\omega \wedge \omega))$.

For $GL(2)$-structures of arbitrary degree n, the curvature, $\nabla(\theta)$, decomposes into $\mathrm{d}\varphi + \frac{1}{2} \langle \varphi, \varphi \rangle_1 = R(\omega \wedge \omega)$ and $\mathrm{d}\lambda = r(\omega \wedge \omega)$. The curvature functions are

$$R : B \to \mathfrak{sl}(2) \otimes \wedge^2(\mathcal{V}_n^*) \text{ and } r : B \to \mathbb{R} \otimes \wedge^2(\mathcal{V}_n^*). \tag{4.18}$$

Moreover, the covariant derivative of the torsion two-form, $\nabla(T(\omega\wedge\omega))$, decomposes as $\nabla(T(\omega \wedge \omega)) = \nabla(T)(\omega \wedge \omega) + 2Q(T,T)(\omega \wedge \omega \wedge \omega)$. The second-order torsion functions are

$$\nabla(T) :\to H^{0,2}(\mathfrak{gl}(2)) \otimes \mathcal{V}_n^* \text{ and } Q : B \to \mathrm{Sym}^2(H^{0,2}(\mathfrak{gl}(2)) \cap (\mathcal{V}_n \otimes \wedge^3 \mathcal{V}_n^*). \quad (4.19)$$

For $n = 4$, the Clebsch–Gordon decomposition provides specific irreducible representations from

$$\begin{aligned} R &= R_0^2 + R_2^2 + R_4^2 + R_4^6 + R_6^6 + R_8^6 \in \mathcal{V}_2 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6), \\ r &= r_2 + r_6 \in \mathcal{V}_0 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6), \\ \nabla T &\in (\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \otimes \mathcal{V}_4, \text{ and} \\ Q &\in \mathrm{Sym}^2(\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \cap (\mathcal{V}_4 \otimes (\mathcal{V}_2 \oplus \mathcal{V}_6)). \end{aligned} \quad (4.20)$$

The Bianchi identity provides relations between the irreducible representations of R, r, ∇T, and Q; however, it is too difficult to analyze until we consider particular systems in Chapters 5 and 6.

5

2-Integrability in Degree Four

In this chapter, we use the Cartan–Kähler theorem to study the differential ideal that describes the existence of many bi-secant surfaces and arrive at a structure theorem for 2-integrable $GL(2)$-structures of degree 4. Unfortunately, these structure equations are not closed under exterior differentiation, but they simplify beautifully in Chapter 6 during the study of 3-integrability.

5.1 Bi-secant Surfaces and 2-Integrability

We want to find the conditions on B that allows any bi-secant plane in $\mathbf{T}M$ to be extended to a bi-secant surface $\Sigma \subset M$. The tangent planes $\mathbf{T}_p\Sigma$ must intersect $\mathbf{C}_p$ in two lines for all $p \in \Sigma$, so $\mathbf{T}_p\Sigma$ is spanned by $(a(p)x+b(p)y)^4$ and $(A(p)x+B(p)y)^4$. Under a $GL(2)$ change of basis in $\mathbf{T}_pM$, we may assume the spanning vectors are x^4 and y^4.

Lifting this problem to B, this is identical to finding integral surfaces of the exterior differential system $\mathcal{I}$ that is differentially generated by 1-forms $\{w^{-2}, w^0, w^2\}$ with independence condition $\Omega = \omega^{-4} \wedge \omega^4 \neq 0$.

We compute the generating two-forms using Equation (4.4) and the unique connection of Corollary 4.2. To make the computation more explicit, we use vector notation as described in Chapter 3:

$$\mathrm{d}\begin{pmatrix}\omega^{-2}\\ \omega^{0}\\ \omega^{2}\end{pmatrix} \equiv \begin{pmatrix}-8\varphi_2 & 0\\ 0 & 0\\ 0 & 8\varphi_{-2}\end{pmatrix} \wedge \begin{pmatrix}\omega^{-4}\\ \omega^{4}\end{pmatrix} + 8\begin{pmatrix}\tau^{-2}\\ \tau^{0}\\ \tau^{2}\end{pmatrix} \omega^{-4}\wedge\omega^{4} \tag{5.1}$$

modulo $\omega^{-2}, \omega^{0}, \omega^{2}$, where

$$\begin{aligned}
\tau^{-2} &= -11520\ T_{10,-2} + 2880\ T_{8,-2} + 288\ T_{6,-2} + 24\ T_{2,-2}\\
\tau^{0} &= -14400\ T_{10,0} + 864\ T_{6,0} - 36\ T_{2,0}\\
\tau^{2} &= -11520\ T_{10,2} - 2880\ T_{8,2} + 288\ T_{6,2} + 24\ T_{2,2}
\end{aligned} \tag{5.2}$$

Because of the independence condition $\Omega \neq 0$, integral elements exist only when the torsion can be absorbed. The torsion component τ^0 can never be absorbed, so integral manifolds exist only when $\tau^0 = 0$. The condition of 2-integrability means that every 2-secant plane is tangent to a 2-secant surface. The $GL(2)$ action is transitive on 2-secant planes in $\mathbf{T}_pM$; therefore, it must be that $\tau^0 = 0$ for every element in the $GL(2)$ orbit of T. Under a $GL(2)$ action, the coordinates of the irreducible representations of T will change, so each irreducible representation that appears in τ^0 must vanish identically. Hence, 2-integrability of M by integral manifolds implies

$$T_{10} = T_6 = T_2 = 0. \tag{5.3}$$

The remaining torsion components, τ^{-2} and τ^2, can be absorbed easily. Let $\pi_1 = -8\varphi_2 - 8\tau^{-2}\ \omega^4$ and $\pi_2 = 8\varphi_{-2} + 8\tau^2\ \omega^{-4}$, so

$$\mathrm{d}\begin{pmatrix}\omega^{-2}\\ \omega^{0}\\ \omega^{2}\end{pmatrix} \equiv \begin{pmatrix}\pi_1 & 0\\ 0 & 0\\ 0 & \pi_2\end{pmatrix} \wedge \begin{pmatrix}\omega^{-4}\\ \omega^{4}\end{pmatrix}, \quad \mod \omega^{-2}, \omega^{0}, \omega^{2}. \tag{5.4}$$

We can now apply Cartan's test to the linear Pfaffian system whose tableau is given in Equation (5.4) [BCG+91] [IL03]. For a generic flag of $\mathbf{T}_pN$ obtained from generic linear combinations of ω^{-4} and ω^4, this tableau has Cartan characters $s_1 = 2$ and $s_2 = 0$. The space of integral elements for the EDS $(\mathcal{I}, \Omega)$ is 2-dimensional, as parametrized by $p_{1,4}$ and $p_{3,-4}$ where $\pi_1 = p_{1,4}\, \omega^4$ and $\pi_2 = p_{3,-4}\, \omega^{-4}$. Therefore Cartan's test indicates that the system in involutive. Because of the use of the Cartan–Kähler theorem here, real-analyticity is required. This discussion proves the following theorem.

Theorem 5.1. *If a $GL(2)$-structure B is 2-integrable, then $T = T_8$ (that is, $T_2 = T_6 = T_{10} = 0$) and bi-secant surfaces are parametrized by two functions of one variable. Conversely, a $GL(2)$-structure B is analytic with $T = T_8$, then B is 2-integrable.*

Note that analyticity is unlikely to be necessary and can probably be weakened to smoothness using more powerful techniques. For 2-integrable $GL(2)$-structures arising from PDEs of hydrodynamic type, the parametrization by two functions of one variable confirms the computation presented in [FHK07]. The necessary condition $T = T_8$ appears to be new.

5.2 Curvature

To find additional necessary conditions, we now examine the Bianchi identity of B. In particular, consider Equation 4.20 under the condition $T = T_8$. In this case, the decompositions of ∇T and Q simplify considerably:

$$\begin{aligned} R &= R^2_0 + R^2_2 + R^2_4 + R^6_4 + R^6_6 + R^6_8 \in \mathcal{V}_2 \otimes (\wedge^2 \mathcal{V}_4^*), \\ r &= r_2 + r_6 \in \mathcal{V}_0 \otimes (\wedge^2 \mathcal{V}_4^*), \\ \nabla T &= S_4 + S_6 + S_8 + S_{10} + S_{12} \in \mathcal{V}_{n+4} \otimes \mathcal{V}_4^*. \\ Q &= Q_4 + Q_8 \in S^2(\mathcal{V}_{n+4}) \cap (\mathcal{V}_4 \otimes \wedge^3 \mathcal{V}_4^*). \end{aligned} \tag{5.5}$$

Expanding the Bianchi identity with these substitutions, we obtain

$$
\begin{aligned}
\nabla(\theta)\wedge\omega &= \langle\nabla(\phi),\omega\rangle_1 + \langle \mathrm{d}\lambda,\omega\rangle_0 \\
&= \langle R(\omega\wedge\omega),\omega\rangle_1 + \langle r(\omega\wedge\omega),\omega\rangle_0 \\
&= + \left\langle \left\langle R^2_0, \langle\omega,\omega\rangle_3\right\rangle_0 + \left\langle R^2_2, \langle\omega,\omega\rangle_3\right\rangle_1 + \left\langle R^2_4, \langle\omega,\omega\rangle_3\right\rangle_2, \omega_1 \right\rangle \\
&\qquad + \left\langle \left\langle R^6_4, \langle\omega,\omega\rangle_1\right\rangle_4 + \left\langle R^6_6, \langle\omega,\omega\rangle_1\right\rangle_5 + \left\langle R^6_6, \langle\omega,\omega\rangle_1\right\rangle_6, \omega \right\rangle_1 \\
&\qquad + \left\langle \left\langle r_2, \langle\omega,\omega\rangle_3\right\rangle_2 + \left\langle r_6, \langle\omega,\omega\rangle_1\right\rangle_6, \omega \right\rangle_0,
\end{aligned}
\tag{5.6}
$$

$$
\left\langle \nabla T_8, \langle\omega,\omega\rangle_1\right\rangle_5 = \langle \langle S_4,\omega\rangle_0 + \langle S_6,\omega\rangle_1 + \langle S_8,\omega\rangle_2 + \langle S_{10},\omega\rangle_3 + \langle S_{12},\omega\rangle_4, \langle\omega,\omega\rangle_1\rangle_5,
\tag{5.7}
$$

and

$$
\begin{aligned}
Q(T\circ T)(\omega\wedge\omega\wedge\omega) = &\left\langle Q_4, \langle\langle\omega,\omega\rangle_A,\omega\rangle_B + \langle\langle\omega,\omega\rangle_A,\omega\rangle_B\right\rangle_C \\
&+ \left\langle Q_8, \langle\langle\omega,\omega\rangle_A,\omega\rangle_B + \langle\langle\omega,\omega\rangle_A,\omega\rangle_B\right\rangle_C.
\end{aligned}
\tag{5.8}
$$

Hence, the Bianchi identity implies relations among the various representations of R, r, S, and Q.

Theorem 5.2. *If a $GL(2)$-structure B of degree 4 is 2-integrable, then $T = T_8$ and these conditions must hold: $S_{10} = 0$, $R^6_8 = 33\ S_8$, $R^6_6 = 45\ S_6$, $r^6_6 = 960\ S_6$ $R^6_4 = -8\ Q_4 + -12\ S_4$, $R^2_4 = 48\ Q_4 + 42\ S_4$, $R^2_2 = 0$, and $r^2_2 = 0$.*

Notice that R^2_0 is the only term from $\nabla(\theta)$ that is still free. That is, 2-integrable $GL(2)$-structures are characterized locally by two functions, $T_8 \in \mathcal{V}_8$ and $R^2_0 \in \mathcal{V}_0$.

6

3-Integrability in Degree Four

In this chapter, we use the Cartan–Kähler theorem to study the differential ideal that describes the existence of many tri-secant 3-folds in a five-dimensional manifold. This leads to a structure theorem for certain 3-integrable $GL(2)$-structures of degree 4 and to the discovery of a local classification of these objects, whose topology is studied in Chapter 8.

6.1 Tri-secant 3-Folds and 3-Integrability

We want to find the conditions on B that allow any tri-secant 3-plane in $\mathbf{T}M$ to be extended to a tri-secant 3-fold $N \subset M$. The tangent spaces $\mathbf{T}_pN$ must intersect $\mathbf{C}_p$ in three independent lines for all $p \in N$. Under a $GL(2)$ change of basis in $\mathbf{T}_pM$ we may assume the spanning vectors are x^4 and y^4 and $(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$. Hence, any vector tangent to the 3-fold looks like $(a+b)\ x^4 + b\ 4x^3y + b\ 6x^2y^2 + b\ 4xy^4 + (b+c)\ y^4$. Lifting this problem to B, these vectors are in the kernel of two 1-forms, $\kappa^{-2} = \omega^{-2} - \omega^0$ and $\kappa^2 = \omega^2 - \omega^0$.

This problem is then identical to finding integral 3-folds of the EDS $\mathcal{I}$ differentially generated by 1-forms $\{\kappa^{-2}, \kappa^2\}$ with independence condition $\Omega = \omega^{-4} \wedge \omega^0 \wedge \omega^4 \neq 0$.

The tableau and torsion for this system are given by

$$\mathrm{d}\begin{pmatrix}\kappa^{-2}\\ \kappa^{2}\end{pmatrix} \equiv \begin{pmatrix}\pi_1 & \pi_3 & 0\\ 0 & -\pi_1-\pi_2-\pi_3 & \pi_2\end{pmatrix}\wedge\begin{pmatrix}\omega^{-4}\\ \omega^{0}\\ \omega^{4}\end{pmatrix} + \tau(\omega,\omega), \tag{6.1}$$

modulo κ^{-2}, κ^2, where $\pi_1 = -2\varphi_2$, $\pi_2 = 2\varphi_{-2}$, and $\pi_3 = 2\varphi_{-2} - 4\varphi_0 + 4\varphi_2$. There are two questions: Is the torsion absorbable under a change of basis, and what is the dimension of the space of integral elements?

If $\hat{\pi}_i = \pi_i - p_{i,a}\omega^a$ is a generic change of basis, then these equations become

$$\begin{aligned}\mathrm{d}\kappa^{-2} &\equiv \pi_1\wedge\omega^{-4} + \pi_3\wedge\omega^0 - \tau^{-2}_{-4,0}\omega^{-4}\wedge\omega^0 - \tau^{-2}_{-4,4}\omega^{-4}\wedge\omega^4 - \tau^{-2}_{0,4}\omega^0\wedge\omega^4\\ &= \hat{\pi}_1\omega^{-4} + \hat{\pi}_3\omega^0 + (-p_{1,0} + p_{3,-4} - \tau^{-2}_{-4,0})\omega^{-4}\wedge\omega^0\\ &\quad + (-p_{1,4} - \tau^{-2}_{-4,4})\omega^{-4}\wedge\omega^4 + (-p_{3,4} - \tau^{-2}_{0,4})\omega^0\wedge\omega^4,\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}\mathrm{d}\kappa^{2} &\equiv -(\pi_1+\pi_2+\pi_3)\wedge\omega^0 + \pi^2\wedge\omega^4 - \tau^{2}_{-4,0}\omega^{-4}\wedge\omega^0 - \tau^{2}_{-4,4}\omega^{-4}\wedge\omega^4 - \tau^{2}_{0,4}\omega^0\wedge\omega^4\\ &= -(\hat{\pi}_1+\hat{\pi}_2+\hat{\pi}_3)\wedge\omega^0 + \hat{\pi}^2\wedge\omega^4 - (\tau^2_{-4,0} + p_{1,-4} + p_{2,-4} + p_{3,-4})\omega^{-4}\wedge\omega^0\\ &\quad + (p_{2,-4} - \tau^2_{-4,4})\omega^{-4}\wedge\omega^4 + (p_{1,4} + p_{2,4} + p_{3,4} + p_{2,0} - \tau^2_{0,4})\omega^0\wedge\omega^4\end{aligned} \tag{6.3}$$

The apparent torsion can be fully absorbed by setting

$$\begin{aligned}p_{1,4} &= -\tau^{-2}_{-4,4}\\ p_{2,-4} &= \tau^{2}_{-4,4}\\ p_{3,4} &= -\tau^{-2}_{0,4}\\ p_{3,-4} &= -p_{1,-4} - \tau^2_{-4,4} - \tau^2_{-4,0}\\ p_{2,4} &= -p_{2,0} - p_{3,4} - p_{1,4} + \tau^2_{0,4} = -p_{2,0} + \tau^{-2}_{0,4} + \tau^{-2}_{-4,4} + \tau^2_{0,4}\\ p_{1,0} &= p_{3,-4} - \tau^{-2}_{-4,4} = -p_{1,-4} - \tau^2_{-4,4} - \tau^2_{-4,0} - \tau^2_{-4,4}.\end{aligned} \tag{6.4}$$

The integral elements are still free up to arbitrary choice of three variables, $p_{1,-4}$, $p_{2,0}$, and $p_{3,0}$. Since the Cartan characters are $s_1 = 2$, $s_2 = 1$, and $s_3 = 0$, but $s_1 + 2s_2 + 3s_3 = 4 \neq 3$, the tableau is not involutive; prolongation is required.

Let $\mathcal{I}^{(1)}$ be the prolonged ideal, which is differentially generated by the forms κ^{-2} and κ^2 along with

$$\begin{aligned}
\eta^1 &= \pi_1 + p_{1,-4}\ \omega^{-4} - (p_{1,-4} + \tau^2_{-4,4} + \tau^2_{-4,0} + \tau^2_{-4,4})\ \omega^0 - \tau^{-2}_{-4,4}\ \omega^4,\\
\eta^2 &= \pi_2 + \tau^2_{-4,4}\ \omega^{-4} + p_{2,0}\ \omega^0 + (-p_{2,0} + \tau^{-2}_{0,4} + \tau^{-2}_{-4,4} + \tau^2_{0,4})\ \omega^4,\ \text{and}\\
\eta^3 &= \pi_3 + (-p_{1,-4} - \tau^2_{-4,4} - \tau^2_{-4,0})\ \omega^{-4} + p_{3,0}\ \omega^0 + -\tau^{-2}_{0,4}\ \omega^4.
\end{aligned} \tag{6.5}$$

After this prolongation, the tableau and torsion are given by

$$\mathrm{d}\begin{pmatrix}\kappa^{-2}\\ \kappa^2\\ \eta^1\\ \eta^2\\ \eta^3\end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \pi_4 & -\pi_4 & 0\\ 0 & \pi_5 & -\pi_5\\ -\pi_4 & \pi_6 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^{-4}\\ \omega^0\\ \omega^4\end{pmatrix} + \tau^{(1)}(\eta\wedge\eta), \tag{6.6}$$

modulo $\kappa^{-2}, \kappa^2, \eta^1, \eta^2, \eta^3$. This tableau has Cartan characters $s_1 = 3$, $s_2 = 0$, and $s_3 = 0$. Applying Cartan's test, $s_1 + 2s_2 + 3s_3 = 3$, which matches the dimension of $\mathcal{V}_3(\mathcal{I})$, so the tableau is involutive. If the torsion vanishes, we may conclude (in the analytic category) that the integral 3-folds locally depend on three functions of one variable.

Before dealing with the torsion, let us examine the characteristic variety for this involutive linear Pfaffian tableau. For a complete overview of the characteristic variety and its powerful properties, refer to [BCG$^+$91, Chapter V]. Examining the Cartan characters for the tableau, it is clear that the complex characteristic variety has dimension 0 and degree 3. More specifically, let E denote a generic integral element of dimension 3. Restricted to E, the tableau implies $\pi_4 = A\omega^{-4} - A\omega^0$, $\pi_5 = B\omega^0 - B\omega^4$, and $\pi_6 = A\omega^{-4} + C\omega^0$ for parameters A, B, C that uniquely define E in $Gr_3(\mathbf{T}B)$. In other words,

$$E = \ker(\{\kappa^{-2}, \kappa^2, \eta^1, \eta^2, \eta^3, \pi_4 - A(\omega^{-4} - \omega^0), \pi_5 - B(\omega^0 - \omega^4), \pi_6 - A\omega^{-4} - C\omega^0\}). \tag{6.7}$$

Consider $P_1 = \ker(\omega^{-4} - \omega^0) \subset E$, where this 1-form has been restricted to E, so

$$P_1 = \ker(\{\kappa^{-2}, \kappa^2, \eta^1, \eta^2, \eta^3, \pi_4, \pi_5 - B\omega^0 + B\omega^4, \pi_6 - (A+C)\omega^0\}) \in Gr_2(\mathbf{T}B). \tag{6.8}$$

In particular, the polar space of P_1 has dimension greater than $\dim E = 3$, so $\omega^{-4} - \omega^0 \in \mathbb{P}E^*$ is in the characteristic variety of E. Similarly, $P_2 = \ker(\omega^0 - \omega^4)$ and $P_3 = \ker(A\omega^{-4} + C\omega^0)$ are in the characteristic variety as well. Hence, the characteristic variety over generic E is given by three distinct points in $\mathbb{P}E^*$. This implies that each 3-dimensional manifold is foliated by three distinct families of 2-dimensional integral manifolds for the EDS $(\mathcal{I}, \Omega)$.

Theorem 6.1. *If a $GL(2)$-structure of degree 5 is 3-integrable, then tri-secant 3-folds locally depend upon three functions of one variable. Each tri-secant 3-fold is foliated by three families of surfaces integral to $\mathcal{I}^{(1)}$.*

To determine sufficient conditions for *existence* of tri-secant 3-folds, we must study the unabsorbable portion of the remaining torsion, $\tau^{(1)}$. Because $\tau^{(1)}$ is the torsion of the prolonged system $\mathcal{I}^{(1)}$, it will involve second-order invariants of the $GL(2)$-structure B that appear in the Bianchi identity for B, Equation (4.20). Since T is *a priori* valued in $H^{0,2}(\mathfrak{gl}(2)) = \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}$, components of any of the following functions may occur in $\tau^{(1)}$:

$$\begin{aligned} R &\in \mathcal{V}_2 \otimes (\wedge^2 \mathcal{V}_4), \\ r &\in \mathcal{V}_0 \otimes (\wedge^2 \mathcal{V}_4), \\ \nabla T &\in (\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \otimes \mathcal{V}_4, \\ Q &\in \mathrm{Sym}^2(\mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8 \oplus \mathcal{V}_{10}) \cap (\mathcal{V}_4 \otimes \wedge^3 \mathcal{V}_4). \end{aligned} \tag{6.9}$$

The vanishing of the unabsorbable portion of $\tau^{(1)}$ will place restrictions on the various irreducible representations appearing in Equation (6.9). The enormity of Q and ∇T makes decomposition of the unabsorbable portion of $\tau^{(1)}$ extremely difficult. Fortunately, we can make a simplifying assumption that is consistent with the motivating PDE theory in Theorem 1.7.

Definition 6.2 (2,3-Integrability)**.** *A $GL(2)$-structure B is said to be* 2,3-integrable *if it is both 2-integrable and 3-integrable.*

By Theorem 5.1, one may equivalently say that B is 2,3-integrable if B is 3-integrable and $T(B) \subset \mathcal{V}_8$. Restating Theorem 6.1 in this context provides the following corollary.

Corollary 6.3. *If a $GL(2)$-structure $B \to M$ of degree 5 is 2,3-integrable, then tri-secant 3-folds in M locally depend upon three functions of one variable. Moreover, any tri-secant submanifold of M is triply foliated by bi-secant surfaces.*

For 2,3-integrable $GL(2)$-structures arising from PDEs of hydrodynamic type as in Theorem 1.7, the parametrization by three functions of one variable confirms the computation presented in [FHK07]. The triple foliation by bi-secant surfaces appears to be new, though it is not surprising based on the description of highest-weight polynomial subspaces of $\mathcal{V}_n$ seen in [CM09].

Let us now examine the existence question for 2,3-integrable $GL(2)$-structures. If B is 2,3-integrable, then the conditions on R, r, S, and Q established in Theorem 5.2 hold. Under this assumption, examination of the unabsorbable portion of $\tau^{(1)}$ and verification of the equations $\mathrm{d}(\mathrm{d}\eta^i) \equiv 0 \mod \mathcal{I}^{(1)}$ together force the following conditions:

$$\begin{aligned} R_0^2 = -2080 \left\langle T, T \right\rangle_8, \ S_4 = \frac{8}{21} \left\langle T, T \right\rangle_6, \ S_6 = 0, \\ S_8 = \frac{8}{77} \left\langle T, T \right\rangle_4, \ S_{12} = \frac{80}{231} \left\langle T, T \right\rangle_2. \end{aligned} \tag{6.10}$$

Notice that both R_0^2 and S (hence $\nabla\theta$ and ∇T) depend only on T, so no new functions arise when differentiating the structure equations for those $GL(2)$-structures that satisfy these integrability conditions.

Theorem 6.4 (2,3-integrable GL(2)-structure equations)**.** *A $GL(2)$-structure B over M^5 is 2,3-integrable if and only if the torsion T of B only takes values in $\mathcal{V}_8$ and the conditions in Equation (6.10) are satisfied. In particular, a 2,3-integrable $GL(2)$-structure B has the following structure equations*

$$
\begin{aligned}
\mathrm{d}\omega &= -\left\langle \varphi, \omega \right\rangle_1 - \left\langle \lambda, \omega \right\rangle_0 + \left\langle T, \left\langle \omega, \omega \right\rangle_1 \right\rangle_5 \\
\mathrm{d}\lambda &= 0 \\
\mathrm{d}\varphi &= -\frac{1}{2} \left\langle \varphi, \varphi \right\rangle_1 + -2080 \left\langle \left\langle T, T \right\rangle_8, \left\langle \omega, \omega \right\rangle_3 \right\rangle_0 + 64 \left\langle \left\langle T, T \right\rangle_6, \left\langle \omega, \omega \right\rangle_3 \right\rangle_2 \\
&\quad - \frac{88}{7} \left\langle \left\langle T, T \right\rangle_6, \left\langle \omega, \omega \right\rangle_1 \right\rangle_4 + \frac{24}{7} \left\langle \left\langle T, T \right\rangle_4, \left\langle \omega, \omega \right\rangle_1 \right\rangle_6 \\
\mathrm{d}T &= J(T) \begin{pmatrix} \omega \\ \lambda \\ \varphi \end{pmatrix}
\end{aligned}
\tag{6.11}
$$

for a 9×9 matrix $J(T)$ whose entries are linear and quadratic polynomials in the coefficients of T, as listed in Appendix A.

Proof. By the discussion leading to Equation (6.10), the conditions are necessary and the structure equations are as given. The conditions are sufficient, since a simple computation of $d^2 \equiv 0$ (in MAPLE) verifies that the structure equations fulfill the properties of Cartan's structure theorem, Theorem 2.12. The torsion $T : B \to \mathcal{V}_8$ plays the role of $h : N \to V$. The matrix $J(T)$ plays the role of the matrix $F(h)$ and defines a singular foliation of $\mathcal{V}_8$. For any value $v \in \mathcal{V}_8$, a local solution manifold can be constructed using the symmetry algebra of the leaf $\mathcal{O}_J(v) \subset \mathcal{V}_8$. $\square$

Actually, there is one minor subtlety here. Not all of the J-leaves in $\mathcal{V}_8$ are connected as claimed in Lemma 2.12. Some of them are seen in Chapter 8 to have two components that correspond to the reflection action $\pm I \in GL(2)$. This discrepancy is easily remedied by considering instead the foliation by J-leaves of an open half-space of $\mathcal{V}_8$. As seen in Equation 4.17, $R^*_{-I}(T) = -T$. Hence, for any $GL(2)$-structure

B, $T(B) \ni v$ implies $T(B) \ni -v$. In particular, the $GL(2)$-orbit of any "half-leaf" $\mathcal{O}_F(v)/(\pm I)$ will include the entire leaf $\mathcal{O}_F(v)$, so for the purpose of classifying $GL(2)$-structures, this subtlety can be safely ignored.

Much more than existence can be gathered from Cartan's structure theorem—it provides local uniqueness as well. To understand the consequences, let us make some observations based on the results of Chapter 2. Since these statements are essentially local, a *connected and pointed* space is useful.

Definition 6.5. *A* connected pointed 3-integrable $GL(2)$-structure*, written* (B, M, p)*, is a 3-integrable $GL(2)$-structure of degree 4, $B \to M \ni p$ with M connected.*

Note again the critically important condition that M must be connected! We can now take full advantage of the local equivalence classes built into Cartan's structure theorem.

Lemma 6.6. *(B, M, p) and $(\hat{B}, \hat{M}, \hat{p})$ admit a local $GL(2)$-equivalence $f : M \to \hat{M}$ with $f(p) = \hat{p}$ if and only if $T(\pi^{-1}(p)) \cap \hat{T}(\hat{\pi}^{-1}(\hat{p})) \neq \emptyset$. That is, the value of T at a single point uniquely defines a local 2,3-integrable $GL(2)$-structure of degree* 4 *up to $GL(2)$-equivalence.*

Proof. Let $U \ni p$ and $\hat{U} \ni \hat{p}$ be neighborhoods such that $f : U \to \hat{U}$ is a diffeomorphism. Then $(f^1)^*(\hat{\omega}) = \omega$ implies, so modulo this relation we have

$$\begin{aligned} (f^1)^*(\mathrm{d}\hat{\omega}) - \mathrm{d}\omega &= (f^1)^*(-\hat{\theta} \wedge \hat{\omega} + \hat{T}(\hat{\omega} \wedge \hat{\omega})) + \theta \wedge \omega - T(\omega \wedge \omega) \\ &= -((f^1)^*\hat{\theta} - \theta) \wedge \omega + ((f^1)^*\hat{T} - T)(\omega \wedge \omega)). \end{aligned} \tag{6.12}$$

In particular, $0 = (f^1)^*\hat{T} - T = (\hat{T} \circ f^1) - T$. Therefore, for any $b \in \pi^{-1}(U)$, $T(b) = \hat{T}(f^1(b))$.

Conversely, suppose $\pi(b) = p$ and $\hat{\pi}(\hat{b}) = \hat{p}$ with $\hat{T}(\hat{b}) = T(b) = v$. That is, (B, M, p) and $(\hat{B}, \hat{M}, \hat{p})$ both represent v. By Lemma 2.15, there are neighborhoods

$\pi^{-1}(U) \ni b$ and $\hat{\pi}^{-1}(\hat{U}) \ni \hat{b}$ and diffeomorphism $F : \pi^{-1}(U) \to \hat{\pi}^{-1}(\hat{U})$ such that F^* preserves the 1-forms ω, λ, φ and the torsion T. In particular, F is a $GL(2)$-equivariant diffeomorphism by Lemma 2.6. $\square$

If we can identify the J-leaves in $\mathcal{V}_8$, then we can locally classify the 2,3-integrable $GL(2)$-structures of degree 4 up to leaf-equivalence. Leaf-equivalence is weaker than classification by $GL(2)$-equivalence, but the obstructions preventing leaf-equivalence from implying $GL(2)$-equivalence are topological, so they cannot be determined with our strictly local methods. Thus, leaf-equivalence is the strongest local notion of equivalence that is actually tractable.

6.2 Algebraic Observations

Now that the importance of $J(T)$ is clear, we proceed to study its remarkable properties. The first clue that $J(T)$ is interesting arises by computing its determinant.

Lemma 6.7. *The determinant of $J(T)$ is a scalar multiple of the discriminant of T.*

Proof. We have an explicit formula for J, as shown in Appendix A, so this is a direct computation (using MAPLE). $\square$

Corollary 6.8. *If T has eight distinct roots, then the components of T give local coordinates on B, and these coordinates descend to M.*

Proof. When T has eight distinct roots, J is invertible, so ω, φ, and λ can be written in terms of $\mathrm{d}T$. Since T is a $GL(2)$-equivariant map on B, T descends to a map $M \to \mathcal{V}_8/GL(2)$. $\square$

The relationship between the polynomial T and the matrix $J(T)$ continues beautifully:

Lemma 6.9. *If T is a nontrivial polynomial with k distinct roots, then the rank of J is $k+1$.*

Proof. This is directly verified by writing $T = (h_1x - g_1y)(h_2x - g_2y)\cdots(h_8x - g_8y)$, imposing multiplicity on the h_i's and g_i's and computing the rank of J directly (using MAPLE). Here are some examples of what appears for low values of k:

Suppose $T = (h_1x - g_1y)^8$. All of the 3×3 minors of J vanish. Of the 1296 2×2 minors, 216 of them are nonzero, all of which are monomials of degree 16 in g_1 and h_1.

Suppose $T = (h_1x - g_1y)^7(h_2x - g_2y)$. All of the 4×4 minors of J vanish. Of the 7056 3×3 minors, 2856 are nonzero, all of which are divisible by $(h_1g_2 - h_2g_1)$. Of these nonzero minors, 2851 minors are divisible by h_1; a different 2851 are divisible by g_1, and 84 are divisible by $h_1g_2 + h_2g_1$. Up to scalar multiples, these are all of the factors that appear.

Suppose $T = (h_1x - g_1y)^6(h_2x - g_2y)^2$. All of the 4×4 minors vanish. 5376 of the 3×3 minors are nonzero, all of which are divisible by $(h_1g_2 - h_2g_1)$.

The computations continue in this way for all possible factorizations of T. More complicated factors appear in the non-zero minors, but all are divisible by a term that is appropriate to keep the existing roots of T distinct. $\square$

Lemma 6.10. *Locally, there is a unique 2,3-integrable $GL(2)$-structure with $T = 0$, and it is a local Lie group of dimension nine.*

Proof. If $T = 0$, then Theorem 6.4 shows that the structure equations are those of a local Lie group, as in Theorem 2.9. $\square$

Theorem 6.4 shows that the singular foliation of $\mathcal{V}_8$ defined by $J(T)$ provides a leaf-classification of 2,3-integrable $GL(2)$-structures of degree 4. After a review of the $GL(2)$ orbits on $\mathcal{V}_8$ in Chapter 7, this classification is identified in Chapter 8.

7

The Binary Octics

This chapter is dedicated to the structure of the space $\mathcal{V}_8$ and the action on it by $GL(2,\mathbb{R})$, which by Theorem 6.4 is helpful in identifying the leaf-equivalence classes of 2,3-integrable $GL(2)$-structures of degree 4. This is closely related to a well-studied problem in algebraic geometry, the moduli space of curves with marked points [HM98]. However, the naive perspective taken in this chapter is sufficient for the local classification of 3-integrable $GL(2)$-structures of degree 4.

Fix $v \in \mathcal{V}_8$. Let $\mathrm{Stab}(v) \subset GL(2,\mathbb{R})$ denote the stabilizer of v. Recall the orbit-stabilizer theorem: $v \cdot GL(2,\mathbb{R}) \cong GL(2,\mathbb{R})/\mathrm{Stab}(v)$, and $v \cdot GL(2,\mathbb{R}) = \hat{v} \cdot GL(2,\mathbb{R})$ implies $\mathrm{Stab}(v)$ is conjugate to $\mathrm{Stab}(\hat{v})$. In particular, this implies that no orbit has dimension greater than $\dim GL(2,\mathbb{R}) = 4$. Since $\dim \mathcal{V}_8 = 9$, even the largest orbits are numerous and fail to be open. These orbits are extraordinarily complicated and poorly understood.

While homogeneous binary polynomials of lesser degree have been carefully studied in [Ell96], [Olv90], and elsewhere, to my knowledge the orbits of $GL(2,\mathbb{R})$ on $\mathcal{V}_8$ have never been enumerated. However, the general problem of describing $\mathcal{V}_8/GL(2,\mathbb{R})$ has been well-studied. Classically, [Ell96] provides a normal form for binary "octavics" and describes some invariants. Using the moving-frame methods of Cartan, [BO00] provides invariants that help distinguish the orbits of $GL(2,\mathbb{C})$ on $\mathcal{V}_8$. These results more-or-less directly apply to the present problem as long as we restrict our attention to $GL(2,\mathbb{R}) \subset GL(2,\mathbb{C})$. Using modern algebro-geometric techniques, [Chu06] (revised in [Chu07]) makes a careful study of real binary octics that are stable in the sense of Mumford's geometric invariant theory and extends the study of complex binary octics seen in [MY93]. For our purposes, these advanced results are mostly unnecessary, but their depth emphasizes our extraordinary luck that the J-leaves are as easily understood as they turn out to be in Chapter 8.

7.1 Linear-Fractional Transformations and Root-Types

Let $\mathcal{P}_8$ denote the 9-dimensional vector space of polynomials of degree at most eight in the variable p. There is an isomorphism $\Phi : \mathcal{V}_8 \to \mathcal{P}_8$ via $f(x,y) \mapsto F(p) = f(p,1)$ with inverse $F(p) \mapsto f(x,y) = F(\frac{x}{y})y^8$. The action of $GL(2,\mathbb{R})$ on $\mathcal{P}_8$ is given by $F(p) \cdot g \mapsto F\left(\frac{\alpha p+\beta}{\gamma p+\delta}\right)(\gamma p+\delta)^8$ where $\alpha\delta - \gamma\beta \neq 0$.

One can directly compute that Φ respects the $GL(2,\mathbb{R})$ actions, so Φ is an isomorphisms between these two 9-dimensional (irreducible) representations of $GL(2,\mathbb{R})$. Therefore, any statement about the orbits or stabilizers of the $GL(2,\mathbb{R})$ action on $\mathcal{P}_8$ has a corresponding statement for $\mathcal{V}_8$.

The advantage of considering $\mathcal{P}_8$ is that we can directly apply the theory of linear-fractional transformations on the Riemann sphere. The action of $GL(2,\mathbb{C})$ on $\mathbb{CP}^1$ is given by $p \mapsto \frac{\alpha p+\beta}{\gamma p+\delta}$ where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ and $\alpha\delta - \gamma\beta \neq 0$. We are concerned with the subgroup $GL(2,\mathbb{R})$ where $\alpha,\beta,\gamma,\delta$ are real. The next four lemmas are easy exercises from elementary complex analysis [Ahl78] [Con78].

Lemma 7.1 (Invertibility of linear-fractional transformations)**.** *Linear-fractional transformations are one-to-one. Hence, $S(z) = S(w)$ if and only if $z = w$.*

Lemma 7.2 (Circles to Circles)**.** *Given z_1, z_2, z_3 distinct in $\mathbb{CP}^1$ and w_1, w_2, w_3 distinct in $\mathbb{CP}^1$, there is a unique linear-fractional transformation S such that $S(z_1) = w_1$, $S(z_2) = w_2$, and $S(z_3) = w_3$.*

Lemma 7.3 (Real linear-fractional transformations)**.** *Any linear-fractional transformation S such that $S(\mathbb{RP}) = \mathbb{RP}$ may be written with real coefficients. Conversely, any real linear-fractional transformation maps $\mathbb{RP}$ to itself.*

Lemma 7.4 (Symmetry Principle)**.** *Let S be a real linear-fractional transformation. Then $S(\overline{z}) = \overline{S(z)}$ for all $z \in \mathbb{C}$.*

Applying these four lemmas, we arrive at a simple and well-known observation that is surprisingly advantageous to our cause.

Theorem 7.5 (Root-types are preserved)**.** *For $f(x, y) \in \mathcal{V}_8$, the multiplicity and complex type of the roots are preserved under the $GL(2, \mathbb{R})$ action.*

Proof. For notational simplicity, we instead work with $\Phi(f) = F \in \mathcal{P}_8$, factored as

$$f = A \prod_{k=1}^{m} (p + g_k)^{r_k}, \qquad r_1 + r_2 + \cdots + r_m = 8 \tag{7.1}$$

where $g_k \in \mathbb{CP}^1$ and $g_k \neq g_j$ for $k \neq j$

Fix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{R})$. Then

$$\begin{aligned} F(p) \cdot g &= (\gamma p + \delta)^8 A \prod_{k=1}^{m} \left(\left(\frac{\alpha p + \beta}{\gamma p + \delta} \right) + g_k \right)^{r_k} \\ &= A \prod_{k=1}^{m} \left((\alpha + g_k \gamma) p + (\beta + g_k \delta) \right)^{r_k} \end{aligned} \tag{7.2}$$

Hence, the root g_k has been moved to $S(g_k) = \frac{\delta g_k + \beta}{\gamma g_k + \alpha} \in \mathbb{CP}^1$. Since $\delta\alpha - \beta\gamma \neq 0$, S is a linear-fractional transformation. Therefore, $g_k \neq g_j$ implies $S(g_k) \neq S(g_j)$, so the number and multiplicity of roots is unchanged. Since the action is by $GL(2,\mathbb{R})$, real roots remain real. If there are complex-conjugate roots, $\overline{g_k} = g_j$, then the Symmetry Principle implies that their images are also complex-conjugate: $\overline{S(g_k)} = S(\overline{g_k}) = S(g_j)$. $\square$

Bearing in mind this theorem, we establish a notation for the *root-types* of a factored polynomial $F(p)$ or $f(x,y)$. Sort the exponents $r_1 \geq r_2 \geq \cdots \geq r_m$ where $r_1 + r_2 + \cdots + r_m = 8$. This partition of 8 is written as $\{r_1, r_2, \ldots, r_m\}$. If g_k and g_{k+1} are complex-conjugate, then we note this by enclosing their exponents in square-braces: $\{r_1, \ldots, [r_k, r_{k+1}], \ldots, r_m\}$. Denote the root-type containing $f(x,y)$ by $[f(x,y)]$. For example, $[x^4(x+iy)^2(x-iy)^2] = \{4, [2,2]\}$. Let $\{0\}$ denote the trivial root-type, the zero polynomial. There are 54 non-trivial root-types in $\mathcal{V}_8$, as represented in Figure 7.1. In Figure 7.1, arrows mean "closure contains," and arrows are transitive. Shaded nodes represent root-types that contain exactly one orbit. Oval nodes represent open root-types; the square node represents the nearly-closed root-type, $\{8\} = (\mathcal{C} \setminus 0) \subset \mathcal{V}_8$. Hexagonal nodes represent root classes that are neither closed nor open. Note that strictly real root-types have two connected components; for example, $\{8\}$ is comprised of the two ends of the cone $\mathcal{C}$.

7.2 Symmetries of Polynomials

The root-types are a coarse view of $\mathcal{V}_8/GL(2,\mathbb{R})$. More detailed results can be obtained through the use of covariants of algebraic curves, which can be derived using another interpretation of Cartan's method of equivalence as seen in [Olv90]. This section is a summary some useful results of Peter Olver and Irina Kogan (née Berchenko) that appear in [Olv90] and [BO00].

A covariant of weight k is a function $C : \mathcal{V}_8 \to \mathbb{R}$ such that $C(f \cdot g) = \det(g)^k C(f)$ for any linear-fractional transformation $g \in GL(2,\mathbb{C})$. The first non-trivial example

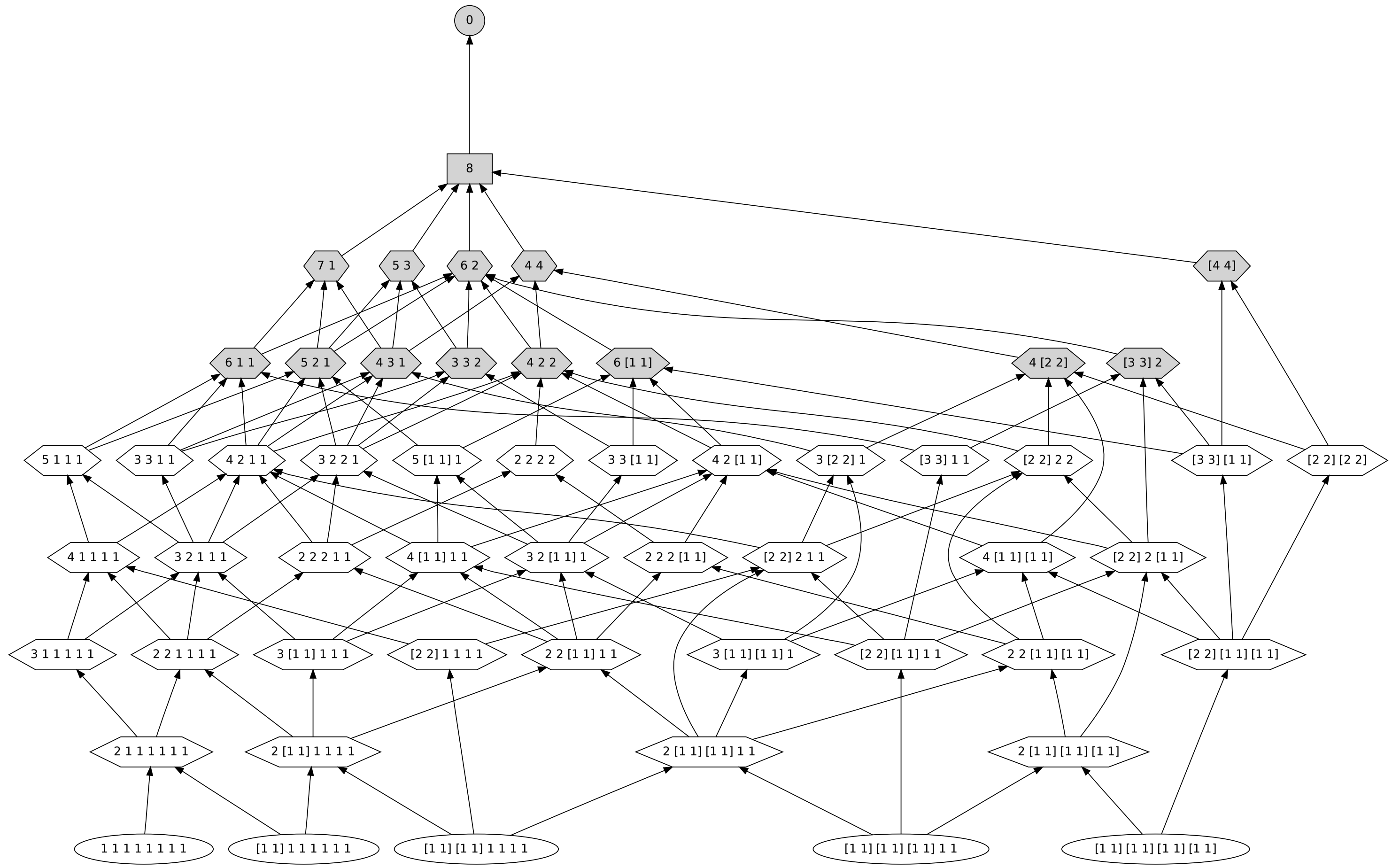

FIGURE 7.1: The stratification of $\mathcal{V}_8$ into the 55 root-types, sorted by dimension. Shaded nodes indicate root-types that contain exactly one orbit. The square node is closed (if 0 is included); the oval nodes are open, and the hexagonal nodes are neither closed nor open.

of a covariant is $H(f) = f_{xx}f_{yy} - (f_{xy})^2$, which has weight 2. Another example, for $F = \Phi(f)$, is

$$T(F) = -n^2(n-1)\left(F^2F''' - 3\frac{n-1}{n}FF'F'' + 2\frac{(n-1)(n-2)}{n^2}(F')^3\right), \qquad (7.3)$$

which is not to be confused with the torsion of a $GL(2)$-structure. The symmetry group of $f \in \mathcal{V}_8$ is the stabilizer group, $\mathrm{Stab}(f) = \{g \in GL(2,\mathbb{C}) \ : \ f \cdot g = f\} \subset GL(2,\mathbb{C})$. When $\mathrm{Stab}(f)$ is a Lie subgroup of $GL(2,\mathbb{C})$, we let $\mathfrak{stab}(f)$ denote its Lie algebra in $\mathfrak{gl}(2,\mathbb{C})$. The use of covariants in [Olv90] and [BO00] provides an efficient algorithm that determines the symmetry group of a given polynomial.

Theorem 7.6 ([Olv90] Theorem 6.4)**.** *Let $f(x,y)$ be a complex binary octic. Then* $\mathrm{Stab}(f)$ *is:*

1. *A 2-parameter group if and only if $H \equiv 0$ if and only if $\Phi(f)$ is in the same $GL(2,\mathbb{C})$ orbit as a constant.*
2. *A 1-parameter group if and only if $H \not\equiv 0$ and T^2 is a constant multiple of H^3 if and only if $\Phi(f)$ is in the same $GL(2,\mathbb{R})$ orbit as a monomial p^k with $k \neq 0, 8$.*
3. *A finite group, otherwise.*

The next theorem is originally in [Bli17], but it is re-proven in [BO00].

Theorem 7.7 ([BO00] Theorem 3.3)**.** *Any finite subgroup of $PGL(2,\mathbb{C})$ is conjugate to one of the following:*

1. *The n-element cyclic group, $\mathcal{Z}_n$, generated by $p \mapsto \omega p$ where ω is a primitive nth root of unity.*
2. *The $2n$-element dihedral group, $\mathcal{D}_n$, generated by adjoining $p \mapsto 1/p$ to $\mathcal{Z}_n$.*

3. The 12 element tetrahedral group $\mathcal{T}$ generated by $\sigma : p \mapsto -p$ and $\tau : p \mapsto \frac{ip+i}{p-1}$

4. The 24-element octahedral group, $\mathcal{O}$, generated by adjoining $p \mapsto ip$ to $\mathcal{T}$

In the case of $PGL(2, \mathbb{R})$, the subgroups $\mathcal{T}$ and $\mathcal{O}$ cannot appear, since they (and their conjugates) must have non-real elements; therefore, the only finite symmetry groups available are $\mathcal{Z}_n$ and $\mathcal{D}_n$.

7.3 Orbits and Stabilizers

7.3.1 One root

If $f(x, y)$ has one real root of multiplicity 8, then $f(x, y) \in \mathcal{C}$, the rational normal cone in $\mathcal{V}_8$. As previously described, $\{8\}$ is the orbit of dimension 2, and the stabilizer of an element is a Lie group of dimension 2. For example the Lie algebra $\mathfrak{stab}(x^8)$ is spanned by $\mathbf{H} - 8\mathbf{I}$ and $\mathbf{X}$. This is root-type $\{8\}$ and is case (1) of Theorem 7.6.

7.3.2 Two roots

If $f(x, y)$ has two distinct roots, then $[f(x, y)]$ is a single $GL(2, \mathbb{R})$-orbit. $F(x) = \Phi(f(x, y))$ has 1-dimension stabilizer as in case (2) of Theorem 7.6, and there is some $g \in GL(2, \mathbb{C})$ such that $F(p) \cdot g = p^k$ for some $1 \leq k \leq 8$. Thus $f(x, y) \cdot g = x^k y^{8-k}$.

When $k = 1, 2, 3$, the we can choose real g such that the roots of F are moved anywhere in $\mathbb{RP}^1$. This is also true when $k = 4$ and both roots are real. If $k = 4$ and the roots of F are complex-conjugates, then complex-conjugacy is preserved by any real linear-fractional transformation. Hence, there are five distinct three-dimensional root-types, enumerated as:

1. $[x^7 y]$, with root-type $\{7, 1\} = \mathcal{Q} \setminus \mathcal{C}$ and $\mathfrak{stab}(x^7 y)$ spanned by $\mathbf{H} - 6\mathbf{I}$;
2. $[x^6 y^2]$, with root-type $\{6, 2\}$ and $\mathfrak{stab}(x^6 y^2)$ spanned by $\mathbf{H} - 4\mathbf{I}$;
3. $[x^5 y^3]$, with root-type $\{5, 3\}$ and $\mathfrak{stab}(x^5 y^3)$ spanned by $\mathbf{H} - 2\mathbf{I}$;

4. $[x^4y^4]$, with root-type $\{4,4\}$ and $\mathfrak{stab}(x^4y^4)$ spanned by $\mathbf{H}$; and

5. $[(x+iy)^4(x-iy)^4]$, with root-type $\{[4,4]\}$ and $\mathfrak{stab}((x+iy)^4(x-iy)^4)$ spanned by $\mathbf{X}+\mathbf{Y}$.

7.3.3 Three roots

With three distinct roots, the possible root-types are $\{6,1,1\}$, $\{6,[1,1]\}$, $\{5,2,1\}$, $\{4,3,1\}$, $\{4,2,2\}$, and $\{4,[2,2]\}$. Each of these root-types is a single orbit, since a linear-fractional transformation allows specification of three points. Using the MAPLE code kindly provided by Olver and Kogan and described in [BO00], we can compute the stabilizer groups for representatives from each orbit:

1. $\mathrm{Stab}(\{6,1,1\})$ is $\mathcal{Z}_2$, generated by $p \mapsto p/(p-1)$ on $\mathcal{P}_8$;

2. $\mathrm{Stab}(\{6,[1,1]\})$ is $\mathcal{Z}_2$, generated by $p \mapsto -p$ on $\mathcal{P}_8$;

3. $\mathrm{Stab}(\{5,2,1\})$ is trivial;

4. $\mathrm{Stab}(\{4,3,1\})$ is trivial;

5. $\mathrm{Stab}(\{4,2,2\})$ is $\mathcal{Z}_2$, generated by $p \mapsto p/(p-1)$ on $\mathcal{P}_8$;

6. $\mathrm{Stab}(\{4,[2,2]\})$ is $\mathcal{Z}_2$, generated by $p \mapsto -p$ on $\mathcal{P}_8$.

7.3.4 Many roots

If a polynomial f has more four or more roots, the stabilizer group $\mathrm{Stab}(f)$ must be a finite group. The root-type $[f]$ is potentially comprised of infinitely many orbits, since each orbit only has dimension three. The space $[f]/GL(2,\mathbb{R})$ may be quite complicated. At least, it can have orbifold singularities; for example if $f_\varepsilon(x,y) = x^2y^2(x+y)^2(x+(2+\varepsilon)y)^2$, then $\mathrm{Stab}(f_0) = \mathcal{D}_4$, but $\mathrm{Stab}(f_\varepsilon)$ is trivial for small $\varepsilon \neq 0$. The moduli of stable real binary octics (those with finite symmetry group) has enjoyed recent exposure in [Chu06] and [Chu07], following the study of the complex case in [MY93]. Regarding the complexity of the space $\mathcal{V}_8/GL(2,\mathbb{R})$, it is shown that the space of stable real binary octics does not even admit the structure of a Riemannian orbifold.

8

Classification of 2,3-Integrable GL(2)-Structures in Degree Four

Lemma 6.6 shows that a 2,3-integrable $GL(2)$-structure of degree 4 is locally determined by the value of T at a point. Cartan's structure theorem shows that J-leaves in $\mathcal{V}_8$ determine certain equivalence classes of connected 2,3-integrable $GL(2)$-structures of degree 4. In this chapter we explicitly describe the leaf-equivalence classes of all 2,3-integrable $GL(2)$-structures of degree 4 by studying $J(T)$. Some of the representatives' structure equations describe solvable Lie algebras and the corresponding open Lie groups can be given reasonable local coordinates.

8.1 Identifying the Leaves

Define an equivalence relation on $\mathcal{V}_8$ by declaring $v \sim w$ if and only if there exists a finite sequence of connected pointed 2,3-integrable $GL(2)$-structures (B_i, M_i, p_i) for $i = 0, \ldots, k$ such that (B_i, M_i, p_i) is $GL(2)$-equivalent to $(B_{i-1}, M_{i-1}, p_{i-1})$ for $i = 1, \ldots, k$ with $v \in T(\pi^{-1}(p_0))$ and $w \in T(\pi^{-1}(p_k))$. In the language of Chapter 2, this equivalence class is $\mathcal{O}_J(v)$, the J-leaf containing v.

Because of Cartan's structure theorem and Theorem 6.6, each J-leaf corresponds to a leaf-equivalence class of 2,3-integrable $GL(2)$-structures of degree 4. Note that the J-leaves must be $GL(2)$-invariant, since the fiber action on a $GL(2)$-structure is an isomorphism. Hence, a J-leaf must be a union of $GL(2)$ orbits in $\mathcal{V}_8$. Lemma 2.12 shows that each J-leaf is a connected manifold, and we can easily compute its tangent space as

$$\mathbf{T}_v\mathcal{O}_J(B) = DT_b(\mathbf{T}_bB) = \operatorname{range} J(v). \tag{8.1}$$

It is possible to identify the J-leaves due to the remarkable matrix J. The fact that $[v]$ is $GL(2)$-invariant and that $\operatorname{rk} J(v) = \dim[v]$ strongly suggest the following theorem.

Theorem 8.1 (Leaf-Equivalence Classes)**.** *The root-types in $\mathcal{V}_8$ are exactly the leaf-equivalence classes of 2,3-integrable $GL(2)$-structures of degree 4. That is,*

$$[v] = \mathcal{O}_J(v),\ \forall v \in \mathcal{V}_8. \tag{8.2}$$

Proof. To prove this theorem, we can compute $\mathbf{T}_v[v]$ and compare it to $\mathbf{T}_v\mathcal{O}_J(v)$. If they are the same for all $v \in \mathcal{V}_8$, then $\mathcal{O}_J(v) = [v]$ for all v.

Fix a root-type $[v]$ and an arbitrary $v \in [v]$. By Equation (8.1), $\mathbf{T}_v\mathcal{O}_J(v) = \operatorname{range} J(v)$. For each column $J_i(v)$ of the matrix $J(v)$, solve $D(v) = J_i(v)$ for $D(v)$, an arbitrary tangent to $[v]$ at v. As it happens, these equations are easily solvable at arbitrary points in all 54 non-trivial root-types and for all columns of J. Because both the leaf-equivalence classes and the root-types partition $\mathcal{V}_8$ by smooth submanifolds, and because $\mathbf{T}_v\mathcal{O}_J(v) = \mathbf{T}_v[v]$ for all $v \in \mathcal{V}_8$, the partitions must be identical.

To illustrate the computations, consider $v = (h_1x + g_1y)^8$, where $[v] = \{8\}$. An arbitrary element of $\mathbf{T}_v[v]$ looks like $D(v) = 8(H_1x + G_1y)(h_1x + g_1y)^7$. Therefore, we must solve $D(v) = J_i(v)$ for H_1 and G_1 in each of the cases $i = 1, \ldots, 9$. As it happens, the first five columns of $J(v)$ are all zero in this case, so the solution is $H_1 = G_1 = 0$. For the remaining four columns, the solutions are elementary to compute as well.

For the other root-types, the computations are similar but somewhat more complicated. All that matters is the fact that they can be solved for arbitrary v. $\square$

8.2 Structure Reduction

To see the "essential" structure of a 2,3-integrable $GL(2)$-structure of degree 4, we now reduce the structure group. Fix (B, M, p) and $v \in T(B)$. Let $\tilde{B}(v) = \{b \in B \ : \ T(b) = v\}$. $\tilde{B}(v)$ is a sub-bundle with fiber $\mathrm{Stab}(v) < GL(2)$. The structure equations for $\tilde{B}(v)$ show no dependence on T (as it has been fixed), so the reduction from $GL(2)$ to $\mathrm{Stab}(v)$ effectively simplifies the structure equations from requiring Cartan's structure theorem to requiring only Lie's third fundamental theorem. After the structures have been reduced, local coordinates may be obtained on $\tilde{B}(v)$ using only the flow-box theorem. Thus, a neighborhood in $\tilde{B}(v)$ may be represented by a local Lie group, and M is locally the symmetric space $\tilde{B}(v)/\mathrm{Stab}(v)$.

To compute the reduced structures, recall the stabilizers computed by [BO00] and summarized in Chapter 7.

8.2.1 One Root

Suppose $v \in T(B)$ lies in the unique 2-dimensional root-type, $\{8\} = \mathcal{C} \setminus 0$. Since $\{8\}$ is itself a $GL(2)$ orbit, $T(B) \ni v'$ for all $v' \in \{8\}$, so we may as well assume that $v = x^8$. $\mathrm{Stab}(x^8)$ is a two-dimensional Lie subgroup and its Lie algebra is spanned by $\mathbf{H} - 8\mathbf{I}$ and $\mathbf{X}$. Notice that $\mathrm{d}T_{-8}(x^8) = \lambda + 16\varphi_0$ and $\mathrm{d}T_{-6}(x^8) = 2\varphi_2$ while $\mathrm{d}T_k = 0$ for $k > -6$. In particular, $\mathrm{d}T$ is a vertical 1-form on B, so T only varies along the $GL(2)$ fiber of B. Therefore, the value of $T/GL(2)$ is locally constant as $[x^8]$ on M. The stabilizer algebra may be computed as $\mathfrak{stab}(x^8) = \ker \mathrm{d}T(x^8)$. In any case, the reduced bundle $\tilde{B}(x^8)$ has a 2-dimensional fiber over M^5. The reduced structure equations are given in Equation (8.3); they describe the structure of a solvable Lie algebra and are integrated and studied in Section 8.3.

$$
\begin{aligned}
\mathrm{d}\omega^{-4} &= 24\ \varphi_0 \wedge \omega^{-4} - 8\ \varphi_{-2} \wedge \omega^{-2} + 2 \cdot 322560\ \omega^0 \wedge \omega^4, \\
\mathrm{d}\omega^{-2} &= 20\ \varphi_0 \wedge \omega^{-2} - 6\ \varphi_{-2} \wedge \omega^0 + 322560\ \omega^2 \wedge \omega^4, \\
\mathrm{d}\omega^{0} &= 16\ \varphi_0 \wedge \omega^{0} - 4\ \varphi_{-2} \wedge \omega^2, \\
\mathrm{d}\omega^{2} &= 12\ \varphi_0 \wedge \omega^{2} - 2\ \varphi_{-2} \wedge \omega^4, \\
\mathrm{d}\omega^{4} &= 8\ \varphi_0 \wedge \omega^{4}, \\
\mathrm{d}\varphi_0 &= 0, \quad \mathrm{d}\varphi_{-2} = 4\ \varphi_0 \wedge \varphi_{-2}.
\end{aligned}
\tag{8.3}
$$

8.2.2 Two Roots

Each of the 3-dimensional root-types is a single $GL(2)$-orbit. Again, $T(B) = [v]$ for any v in the root-type, so an arbitrary representative v may be chosen for any (B, M, p). Each v has a 1-dimensional stabilizer, which is a Lie subgroup of $GL(2)$. The corresponding Lie algebras have already been computed in Chapter 7, but it worthwhile to examine the nature of $J(v)$ in each case. In all cases, $\mathrm{d}T$ has both vertical and semi-basic components, so the embedding of the stabilizer fiber group varies over M. The reduced structure $\tilde{B}(v)$ is a 6-dimensional Lie group.

The root-type $\{7, 1\} = \mathcal{Q}\backslash\mathcal{C}$ gives the tangent developable of $\mathcal{C}$ and is represented by $v = x^7 y$. The stabilizer algebra can be described as

$$
\ker \mathrm{d}T(v) = \ker\{-100800\omega^4 - \varphi^{-2}, \lambda + 12\varphi_0, \varphi^2\}. \tag{8.4}
$$

Imposing these conditions, the reduced structure equations are

$$
\begin{aligned}
\mathrm{d}\omega^{-4} &= -20\ \omega^{-4} \wedge \varphi_0 - 967680\ \omega^{-2} \wedge \omega^4 - 322560\ \omega^0 \wedge \omega^2,\\
\mathrm{d}\omega^{-2} &= -16\ \omega^{-2} \wedge \varphi_0 - 645120\ \omega^0 \wedge \omega^4,\\
\mathrm{d}\omega^{0} &= -12\ \omega^{0} \wedge \varphi_0 - 322560\ \omega^2 \wedge \omega^4,\\
\mathrm{d}\omega^{2} &= -8\ \omega^{2} \wedge \varphi_0,\\
\mathrm{d}\omega^{4} &= -4\ \omega^{4} \wedge \varphi_0,\\
\mathrm{d}\varphi_0 &= 0.
\end{aligned}
\tag{8.5}
$$

The root-type $\{6, 2\}$ is represented by $v = x^6y^2$. The stabilizer algebra can be described as

$$
\ker \mathrm{d}T(v) = \ker\{28800\omega^2 - 1/2\varphi_{-2}, -28800\omega^4 + 1/4\lambda + 2\varphi_0, \varphi_2\}. \tag{8.6}
$$

Imposing these conditions, the reduced structure equations are

$$
\begin{aligned}
\mathrm{d}\omega^{-4} &= -16\ \omega^{-4} \wedge \varphi_0 + 138240\ \omega^{-4} \wedge \omega^4 + 645120\ \omega^{-2} \wedge \omega^2,\\
\mathrm{d}\omega^{-2} &= -12\ \omega^{-2} \wedge \varphi_0 + 103680\ \omega^{-2} \wedge \omega^4 + 322560\ \omega^0 \wedge \omega^2,\\
\mathrm{d}\omega^{0} &= -8\ \omega^{0} \wedge \varphi_0 + 69120\ \omega^0 \wedge \omega^4,\\
\mathrm{d}\omega^{2} &= -4\ \omega^{2} \wedge \varphi_0 + 34560\ \omega^2 \wedge \omega^4,\\
\mathrm{d}\omega^{4} &= 0, \quad d\varphi_0 = 0.
\end{aligned}
\tag{8.7}
$$

The root-type $\{5, 3\}$ is represented by $v = x^5y^3$. The stabilizer algebra can be described as

$$
\ker \mathrm{d}T(v) = \ker\{-43200\omega^0 - 3/2\varphi_{-2}, 28800\omega^2 + 1/8\lambda + 1/2\varphi_0, -2880\omega^4 + \varphi_2\}. \tag{8.8}
$$

Imposing these conditions, the reduced structure equations are

$$\begin{aligned}
\mathrm{d}\omega^{-4} &= -12\ \omega^{-4}\wedge\varphi_0 - 276480\ \omega^{-4}\wedge\omega^2 - 322560\ \omega^{-2}\wedge\omega^0,\\
\mathrm{d}\omega^{-2} &= -8\ \omega^{-2}\wedge\varphi_0 - 184320\ \omega^{-2}\wedge\omega^2,\\
\mathrm{d}\omega^{0} &= -4\ \omega^{0}\wedge\varphi_0 - 92160\ \omega^{0}\wedge\omega^2,\\
\mathrm{d}\omega^{4} &= 4\ \omega^{4}\wedge\varphi_0 + 230400\ \omega^{2}\wedge\omega^4,\\
\mathrm{d}\omega^{2} &= 0, \quad d\varphi_0 = 0.
\end{aligned} \tag{8.9}$$

The root-type $\{4,4\}$ is represented by $v = x^4y^4$. The stabilizer algebra can be described as

$$\ker \mathrm{d}T(v) = \ker\{11520\omega^{-2} - \varphi_{-2}, -276480\omega^0 + \lambda, 11520\omega^2 + \varphi_2\}. \tag{8.10}$$

Imposing these conditions, the reduced structure equations are

$$\begin{aligned}
\mathrm{d}\omega^{-4} &= 322560\ \omega^{-4}\wedge\omega^0 - 8\ \omega^{-4}\wedge\varphi_0,\\
\mathrm{d}\omega^{-2} &= 161280\ \omega^{-2}\wedge\omega^0 - 4\ \omega^{-2}\wedge\varphi_0,\\
\mathrm{d}\omega^{2} &= -161280\ \omega^{0}\wedge\omega^2 + 4\ \omega^{2}\wedge\varphi_0,\\
\mathrm{d}\omega^{4} &= -322560\ \omega^{0}\wedge\omega^4 + 8\ \omega^{4}\wedge\varphi_0,\\
\mathrm{d}\omega^{0} &= 0, \quad d\varphi_0 = 0.
\end{aligned} \tag{8.11}$$

The root-type $\{[4,4]\}$ is represented by $v = (x^2+y^2)^4$. The stabilizer algebra can be described as

$$\ker \mathrm{d}T(v) = \ker \begin{Bmatrix} \omega^{-4} - \omega^4 - 1/46080\varphi_0, \\ \omega^{-2} + \omega^2 + 1/184320\varphi_{-2} - 1/184320\varphi_2, \\ \omega^0 + \omega^4 + 1/2211840\lambda + 1/92160\varphi_0. \end{Bmatrix} \tag{8.12}$$

Imposing these conditions, the reduced structure equations are surprisingly complicated:

$$
\begin{aligned}
d\omega^{-4} &= -1935360\ \omega^{-4} \wedge \omega^{0} - 645120\ \omega^{-4} \wedge \omega^{4} + 8\ \omega^{-2} \wedge \varphi_{-2} \\
&\quad + 737280\ \omega^{-2} \wedge \omega^{2} + 645120\ \omega^{0} \wedge \omega^{4}, \\
d\omega^{-2} &= 783360\ \omega^{-4} \wedge \omega^{-2} - 2488320\ \omega^{-2} \wedge \omega^{0} - 967680\ \omega^{-2} \wedge \omega^{4} - 2\ \omega^{-4} \wedge \varphi_{-2} \\
&\quad - 506880\ \omega^{-4} \wedge \omega^{2} + 6\ \omega^{0} \wedge \varphi_{-2} + 322560\ \omega^{2} \wedge \omega^{4} - 92160\ \omega^{0} \wedge \omega^{2}, \\
d\omega^{0} &= 1290240\ \omega^{-4} \wedge \omega^{0} - 1290240\ \omega^{0} \wedge \omega^{4} - 4\ \omega^{-2} \wedge \varphi_{-2} \\
&\quad - 737280\ \omega^{-2} \wedge \omega^{2} + 4\ \omega^{2} \wedge \varphi_{-2}, \\
d\omega^{2} &= 967680\ \omega^{-4} \wedge \omega^{2} + 1382400\ \omega^{0} \wedge \omega^{2} - 1152000\ \omega^{2} \wedge \omega^{4} \\
&\quad + 2\ \omega^{4} \wedge \varphi_{-2} - 6\ \omega^{0} \wedge \varphi_{-2} + 1198080\ \omega^{-2} \wedge \omega^{0} \\
&\quad + 138240\ \omega^{-2} \wedge \omega^{4} - 322560\ \omega^{-4} \wedge \omega^{-2}, \\
d\omega^{4} &= 645120\ \omega^{-4} \wedge \omega^{4} + 1935360\ \omega^{0} \wedge \omega^{4} - 8\ \omega^{2} \wedge \varphi_{-2} + 737280\ \omega^{-2} \wedge \omega^{2} \\
&\quad - 645120\ \omega^{-4} \wedge \omega^{0}, \\
d\ \varphi_{-2} &= 184320\ \omega^{-4} \wedge \varphi_{-2} - 184320\ \omega^{4} \wedge \varphi_{-2} - 42467328000\ \omega^{-4} \wedge \omega^{-2} \\
&\quad + 118908518400\ \omega^{-2} \wedge \omega^{0} - 118908518400\ \omega^{0} \wedge \omega^{2} \\
&\quad - 42467328000\ \omega^{-4} \wedge \omega^{2} + 76441190400\ \omega^{-2} \wedge \omega^{4} + 76441190400\ \omega^{2} \wedge \omega^{4}.
\end{aligned}
\tag{8.13}
$$

8.2.3 Three Roots

Each of the 4-dimensional root-types is a single $GL(2)$-orbit, so $T(B) = [v]$ for any v in the root-type, so an arbitrary representative v may be chosen for any (B, M, p). Each v has a finite stabilizer that appears in Theorem 7.7. Hence, $\tilde{B}(v)$ is a finite cover of M^5. The fiber of the finite cover can be determined using the algorithms described in [BO00] and implemented in MAPLE, but the relations that compute to the pull-back of (φ, λ) in terms of the semi-basic form ω again arise by examining $0 = dT(v)$.

The root-type $\{6, 1, 1\}$ is represented by $v = x^6 y(x - y)$. The image under pull-back of (φ, λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= -57600\ \omega^2 - 100800\ \omega^4,\\
\varphi_0 &= 72000\ \omega^4,\\
\varphi_2 &= 0,\\
\lambda &= -691200\ \omega^4.
\end{aligned} \tag{8.14}$$

The root-type $\{5,2,1\}$ is represented by $v = x^5y^2(x-y)$. The image under pull-back of (φ, λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= 28800\ \omega^0 + 57600\ \omega^2,\\
\varphi_0 &= -57600\ \omega^2 + 8640\ \omega^4,\\
\varphi_2 &= -2880\ \omega^4,\\
\lambda &= 46080\ \omega^4 + 460800\ \omega^2.
\end{aligned} \tag{8.15}$$

The root-type $\{4,3,1\}$ is represented by $v = x^4y^3(x-y)$. The image under pull-back of (φ, λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= -11520\ \omega^{-2} - 28800\ \omega^0,\\
\varphi_0 &= 40320\ \omega^0 - 23040\ \omega^2,\\
\varphi_2 &= 11520\ \omega^2 + 2880\ \omega^4,\\
\lambda &= -276480\ \omega^0 - 138240\ \omega^2.
\end{aligned} \tag{8.16}$$

The root-type $\{3,3,2\}$ is represented by $v = x^3y^3(x-y)^2$. The image under pull-back of (φ, λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= -2880\ \omega^{-4} - 23040\ \omega^{-2} - 28800\ \omega^0,\\
\varphi_0 &= 23040\ \omega^{-2} - 23040\ \omega^2,\\
\varphi_2 &= 28800\ \omega^0 + 23040\ \omega^2 + 2880\ \omega^4,\\
\lambda &= -138240\ \omega^{-2} - 230400\ \omega^0 - 138240\ \omega^2.
\end{aligned} \tag{8.17}$$

The root-type $\{4,2,2\}$ is represented by $v = x^4y^2(x-y)^2$. The image under pull-back of (φ,λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= 11520\ \omega^{-2} + 57600\ \omega^0 + 57600\ \omega^2,\\
\varphi_0 &= -40320\ \omega^0 - 34560\ \omega^2 + 8640\ \omega^4,\\
\varphi_2 &= -11520\ \omega^2 - 5760\ \omega^4,\\
\lambda &= 276480\ \omega^0 + 46080\ \omega^4 + 276480\ \omega^2.
\end{aligned} \tag{8.18}$$

The root-type $\{6,[1,1]\}$ is represented by $v = x^6(x^2+y^2)$. The image under pull-back of (φ,λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= 57600\ \omega^2,\\
\varphi_0 &= -72000\ \omega^4,\\
\varphi_2 &= 0,\\
\lambda &= 691200\ \omega^4.
\end{aligned} \tag{8.19}$$

The root-type $\{4,[2,2]\}$ is represented by $v = x^4(x^2+y^2)^2$ The image under pull-back of (φ,λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= 11520\ \omega^{-2} - 46080\ \omega^2,\\
\varphi_0 &= -40320\ \omega^0 - 63360\ \omega^4,\\
\varphi_2 &= -11520\ \omega^2,\\
\lambda &= 276480\ \omega^0 + 92160\ \omega^4.
\end{aligned} \tag{8.20}$$

The root-type $\{[3,3],2\}$ is represented by $v = x^2(x^2+y^2)^3$ The image under pull-back of (φ,λ) is computed by the relations

$$\begin{aligned}
\varphi_{-2} &= -126720\ \omega^{-2} - 149760\ \omega^2,\\
\varphi_0 &= -8640\ \omega^{-4} - 40320\ \omega^0 - 54720\ \omega^4,\\
\varphi_2 &= -57600\ \omega^{-2} - 34560\ \omega^2,\\
\lambda &= 46080\ \omega^{-4} - 460800\ \omega^0 - 506880\ \omega^4.
\end{aligned} \tag{8.21}$$

8.2.4 Many Roots

Suppose $\dim[v] \geq 4$; Theorem 7.6 implies $\mathrm{Stab}(v)$ is a finite subgroup that appears in Theorem 7.7. Hence, $\tilde{B}(v)$ is a finite cover of M^5. The fiber of the finite cover can be determined using the algorithms described in [BO00] and implemented in Maple. However, there is *a priori* no reason to believe that $T(B) \subset [v]$ implies $T(B) = [v]$. In fact, we see in Section 8.4 that B with $T(B) = [v]$ fails to exist even for the reasonably simple case of $[v] = \{2, 2, 2, 2\}$.

8.3 Structure Integration

By integrating the reduced structure equations, we can find reasonable local coordinates in which to write the right-invariant co-frame $(\omega, \lambda, \varphi)$. With the co-frame written in local coordinates, we can find explicit formulas for the distribution of rational normal cones and the tri-secant 3-folds in local coordinates on M.

8.3.1 One Root

Assume (B, M, p) has $T(B) = \{8\} = \mathcal{C} \setminus 0$. The reduced structure equations are given in Equation (8.3). These are the structure equations for a solvable Lie algebra, and the equations can be explicitly integrated "by quadrature." In particular, $\tilde{B}(x^8)$ has coordinates $(\xi^{-4}, \xi^{-2}, \xi^0, \xi^2, \xi^4, a, b)$ with $N = 322560$ such that

$$
\begin{aligned}
\varphi_0 &= a^{-1}\ \mathrm{d}a \\
\varphi_{-2} &= a^4\ \mathrm{d}b, \\
\omega^4 &= a^8\ \mathrm{d}\xi^4, \\
\omega^2 &= a^{12}\left(\mathrm{d}\xi^2 - 2b\ \mathrm{d}\xi^4\right), \\
\omega^0 &= a^{16}\left(\mathrm{d}\xi^0 - 4b\ \mathrm{d}\xi^2 + 4b^2\ \mathrm{d}\xi^4\right), \\
\omega^{-2} &= a^{20}\left(\mathrm{d}\xi^{-2} - 6b\ \mathrm{d}\xi^0 + 12b^2\ \mathrm{d}\xi^2 - 8b^3\ \mathrm{d}\xi^4 - N\xi^4\ \mathrm{d}\xi^2\right), \\
\omega^{-4} &= a^{24}\big(\mathrm{d}\xi^{-4} - 8b\ \mathrm{d}\xi^{-2} + 24b^2\ \mathrm{d}\xi^0 - 32b^3\ \mathrm{d}\xi^2 + 16b^4\ \mathrm{d}\xi^4 + \\
&\qquad 8N\xi^4 b\ \mathrm{d}\xi^2 - 2N\xi^4\ \mathrm{d}\xi^0\big).
\end{aligned}
\tag{8.22}
$$

Now we can reconstruct the distribution of rational normal cones, $\mathbf{C} \subset \mathbf{T}M$, and begin to understand the "most symmetric" $GL(2)$-structure in terms of the local coordinate ξ. Recall that $v \in \mathbf{C}_p \subset \mathbf{T}_pM$ if and only if $u_p(v) \in \mathcal{C} \subset \mathcal{V}_4$ for every $u \in B_p$. Hence, $v \in \mathbf{C}_p$ if and only if for any $\mathbf{w} \in \mathbf{T}_uB$ with $\pi(u) = p$ and $\pi_*(\mathbf{w}) = v$ we have $\omega_u(\mathbf{w}) = u \circ \pi_*(\mathbf{w}) \in \mathcal{C} \subset \mathcal{V}_4$. The tautological form pulls-back to $\tilde{B}(x^8)$, so there is a natural lift of $\mathbf{C}$ to $\mathbf{T}\tilde{B}(x^8)$. The condition for $\mathbf{w} \in \mathbf{C}$ is

$$
\begin{cases}
0 = \omega^{-4}(\mathbf{w})\omega^0(\mathbf{w}) - \omega^{-2}(\mathbf{w})\omega^{-2}(\mathbf{w}) \\
0 = \omega^{-4}(\mathbf{w})\omega^2(\mathbf{w}) - \omega^{-2}(\mathbf{w})\omega^0(\mathbf{w}) \\
0 = \omega^{-4}(\mathbf{w})\omega^4(\mathbf{w}) - \omega^{-2}(\mathbf{w})\omega^2(\mathbf{w}) \\
0 = \omega^{-2}(\mathbf{w})\omega^2(\mathbf{w}) - \omega^0(\mathbf{w})\omega^0(\mathbf{w}) \\
0 = \omega^{-2}(\mathbf{w})\omega^4(\mathbf{w}) - \omega^0(\mathbf{w})\omega^2(\mathbf{w}) \\
0 = \omega^0(\mathbf{w})\omega^4(\mathbf{w}) - \omega^2(\mathbf{w})\omega^2(\mathbf{w})
\end{cases}
\tag{8.23}
$$

Writing $\mathbf{w} = \mathbf{w}_a\frac{\partial}{\partial a} + \mathbf{w}_b\frac{\partial}{\partial b} + \mathbf{w}_{-4}\frac{\partial}{\partial \xi^{-4}} + \mathbf{w}_{-2}\frac{\partial}{\partial \xi^{-2}} + \mathbf{w}_0\frac{\partial}{\partial \xi^0} + \mathbf{w}_2\frac{\partial}{\partial \xi^2} + \mathbf{w}_4\frac{\partial}{\partial \xi^4}$, Equation (8.23) admits several types of solutions:

$$
\{\mathbf{w}^{-2} = 0, \mathbf{w}^0 = 0, \mathbf{w}^2 = 0, \mathbf{w}^4 = 0\}; \tag{8.24}
$$

$$\left\{\begin{array}{l}\mathbf{w}^{-4} = 32\mathbf{w}^2 b^3 + 8N\mathbf{w}^2\xi^4 b,\\ \mathbf{w}^{-2} = 12\mathbf{w}^2 b^2 + N\mathbf{w}^2\xi^4,\\ \mathbf{w}^0 = 4\mathbf{w}^2 b,\\ \mathbf{w}^4 = 0\end{array}\right\}; \tag{8.25}$$

$$\left\{\begin{array}{l}\mathbf{w}^{-4} = (\mathbf{w}^2)^2(2N(\mathbf{w}^4)^2\xi^4 + (\mathbf{w}^2)^2)/(\mathbf{w}^4)^3,\\ \mathbf{w}^{-2} = \mathbf{w}^2(N(\mathbf{w}^4)^2\xi^4 + (\mathbf{w}^2)^2)/(\mathbf{w}^4)^2,\\ \mathbf{w}^0 = (\mathbf{w}^2)^2/\mathbf{w}^4\end{array}\right\};\ \text{and} \tag{8.26}$$

$$\left\{\begin{array}{l}\mathbf{w}^{-4} = -16b\left(-\frac{N}{2}\mathbf{w}^2\xi^4 + 3\mathbf{w}^4 b^3 - 2\mathbf{w}^2 b^2 + \frac{N}{2}\mathbf{w}^4\xi^4 b\right),\\ \mathbf{w}^{-2} = N\mathbf{w}^2\xi^4 - 16\mathbf{w}^4 b^3 + 12\mathbf{w}^2 b^2,\\ \mathbf{w}^0 = -4\mathbf{w}^4 b^2 + 4\mathbf{w}^2 b\end{array}\right\}. \tag{8.27}$$

These equations essentially construct a distribution of rational-normal cones $\mathbf{C}$ over a contractible manifold of dimension 5 that corresponds to *the* 3-integrable $GL(2)$-structure of type $\{8\}$.

8.3.2 Two roots

Consider now $\mathcal{Q} \setminus \mathcal{C} = \{7, 1\} = [x^7 y]$. Integrating the reduced structure equations given in Equation (8.5), we obtain local coordinates $(\xi^{-4}, \xi^{-2}, \xi^0, \xi^2, \xi^4, a)$ on $\tilde{B}(x^7 y)$ given by

$$\begin{aligned}\varphi_0 &= a^{-1}\mathrm{d}a,\\ \omega^4 &= a^4\mathrm{d}\xi^4,\\ \omega^2 &= a^8\mathrm{d}\xi^2,\\ \omega^0 &= a^{12}\left(\mathrm{d}\xi^0 + N\xi^4\mathrm{d}\xi^2\right),\\ \omega^{-2} &= a^{16}\left(\mathrm{d}\xi^{-2} + 2N\xi^4\mathrm{d}\xi^0 - N^2\xi^2\mathrm{d}(\xi^4\xi^4)\right),\\ \omega^{-4} &= a^{20}\left(\mathrm{d}\xi^{-4} + 3N\xi^4\mathrm{d}\xi^{-2} + N\xi^2\mathrm{d}\xi^0 - 3N^2\xi^0\mathrm{d}(\xi^4\xi^4)\right).\end{aligned} \tag{8.28}$$

To construct the distribution of rational normal cones, $\mathbf{C} \subset \mathbf{T}\tilde{B}(x^7 y)$, we again use Equation (8.23) to obtain several types of solution vector $\mathbf{w} \in \mathbf{C}$:

$$\{\mathbf{w}^{-2} = 0, \mathbf{w}^0 = 0, \mathbf{w}^2 = 0, \mathbf{w}^4 = 0\}; \tag{8.29}$$

$$\left.\begin{cases} \mathbf{w}^{-4} = N^2\xi^2\xi^4\mathbf{w}^2 - 6N^3(\xi^4)^3\mathbf{w}^2 \\ \mathbf{w}^{-2} = 2N^2(\xi^4)^2\mathbf{w}^2 \\ \mathbf{w}^0 = -N\mathbf{w}^2\xi^4 \\ \mathbf{w}^4 = 0 \end{cases}\right\}; \tag{8.30}$$

$$\left.\begin{cases} \mathbf{w}^{-4} = -(-6N^2(\mathbf{w}^4)^4\xi^0\xi^4 - N^2\mathbf{w}^2(\mathbf{w}^4)^3\xi^2\xi^4 \\ \quad +N(\mathbf{w}^4)^2\xi^2(\mathbf{w}^2)^2 + 6N^3(\mathbf{w}^4)^4(\xi^4)^2\xi^2 + 6N^3(\mathbf{w}^4)^3(\xi^4)^3\mathbf{w}^2 \\ \quad -6N^2(\mathbf{w}^4)^2(\xi^4)^2(\mathbf{w}^2)^2 + 3N\mathbf{w}^4\xi^4(\mathbf{w}^2)^3 - (\mathbf{w}^2)^4)/(\mathbf{w}^4)^3 \\ \mathbf{w}^{-2} = (2N^2(\mathbf{w}^4)^3\xi^2\xi^4 + 2N^2(\mathbf{w}^4)^2(\xi^4)^2\mathbf{w}^2 - 2N(\mathbf{w}^2)^2\mathbf{w}^4\xi^4 \\ \quad +(\mathbf{w}^2)^3)/(\mathbf{w}^4)^2 \\ \mathbf{w}^0 = -\mathbf{w}^2(N\mathbf{w}^4\xi^4 - \mathbf{w}^2)/\mathbf{w}^4 \end{cases}\right\}; \text{ and} \tag{8.31}$$

$$\left.\begin{cases} \mathbf{w}^{-4} = -N^2\xi^4(-6\mathbf{w}^4\xi^0 - \xi^2\mathbf{w}^2 + 6N\mathbf{w}^4\xi^2\xi^4 + 6N(\xi^4)^2\mathbf{w}^2) \\ \mathbf{w}^{-2} = 2N^2\xi^4(\mathbf{w}^4\xi^2 + \mathbf{w}^2\xi^4) \\ \mathbf{w}^0 = -N\mathbf{w}^2\xi^4 \end{cases}\right\}. \tag{8.32}$$

Again, these equations essentially construct a distribution of rational-normal cones $\mathbf{C}$ over a contractible manifold of dimension 5 that corresponds to *the* 3-integrable $GL(2)$-structure of type $\{7, 1\}$.

With the structure equations provided in Section 8.2 for the other root-types of dimension three, the distribution of rational normal cones can similarly be constructed there.

8.3.3 Many Roots

This process can be repeated for the higher root-types as well, assuming the reduced structure equations are those of a solvable Lie algebra. If the Lie algebra is not solvable, then integrating "by quadrature" in the sort of triangular form seen in Equation 8.22 is not possible, but less natural coordinates could still be found. When the reduced bundle $\tilde{B}$ has finite fiber, the bundle is a finite cover of M; hence, the local coordinates on $\tilde{B}$ are also local coordinates on M.

If the goal is to write the canonical co-frame in terms of local coordinates on (B, M, p), then this is trivial for the five open root-types. Since $9 = \dim \mathcal{O}_J(T(B))$,

$J(T)$ is invertible, so the vector components of T provide local coordinates on B.

8.4 Lack of Generating Examples

Given a root-type $[v]$, does there exist (B, M, p) such that $T(B) = \mathcal{O}_J(v) = [v]$? Certainly the answer is "yes" for root-types with three or fewer distinct roots (the shaded nodes in Figure 7.1); each of these root-types is a single $GL(2)$-orbit, so $T(\pi^{-1}(p))$ maps onto the entire root-type for any $p \in M$.

For the higher-dimensional root-types, the answer is probably "no." Consider the root-type $\{2,2,2,2\}$, which is arguably the simplest of the multi-orbit root-types. Recall the observation from Section 7.3.4 that $\{2,2,2,2\}/GL(2,\mathbb{R})$ has an orbifold singularity at $x^2y^2(x+y)^2(x+2y)^2$. If the submersion $T : B \to \{2,2,2,2\}$ were onto, then the induced submersion $T/GL(2) : M \to \{2,2,2,2\}/GL(2)$ would also be onto, thus ill-defined. Topologically, this root-type retracts onto the space of 4 (distinct) marked points on $\mathbb{RP}^1$, modulo the action of $PGL(2,\mathbb{R})$. To prove that the answer is definitely "no" for all higher root-types, one would need to find similar orbifold singularities in each of the remaining 39 root-types.

8.5 The Classification

The 55 leaf-equivalence classes of 2,3-integrable $GL(2)$ structures of degree 4 are represented in Figure 8.1. The tree is isomorphic to Figure 7.1, since Theorem 8.1 proves that the leaf-equivalence classes are the root-types. In Figure 8.1 arrows mean "closure contains," and arrows are transitive. The pentagonal nodes represent classes that reduce to structures of total dimension 5 with 0-dimensional fiber. The hexagonal nodes represent classes that reduce to structures of total dimension 6 with 1-dimensional fiber. The heptagonal node represents the class that reduces to a structure of total dimension 7 with 2-dimensional fiber. The circular node is the trivial root-type, which corresponds to the unique flat structure. The shaded nodes represent those that are integrated to a local Lie group in Section 8.3.

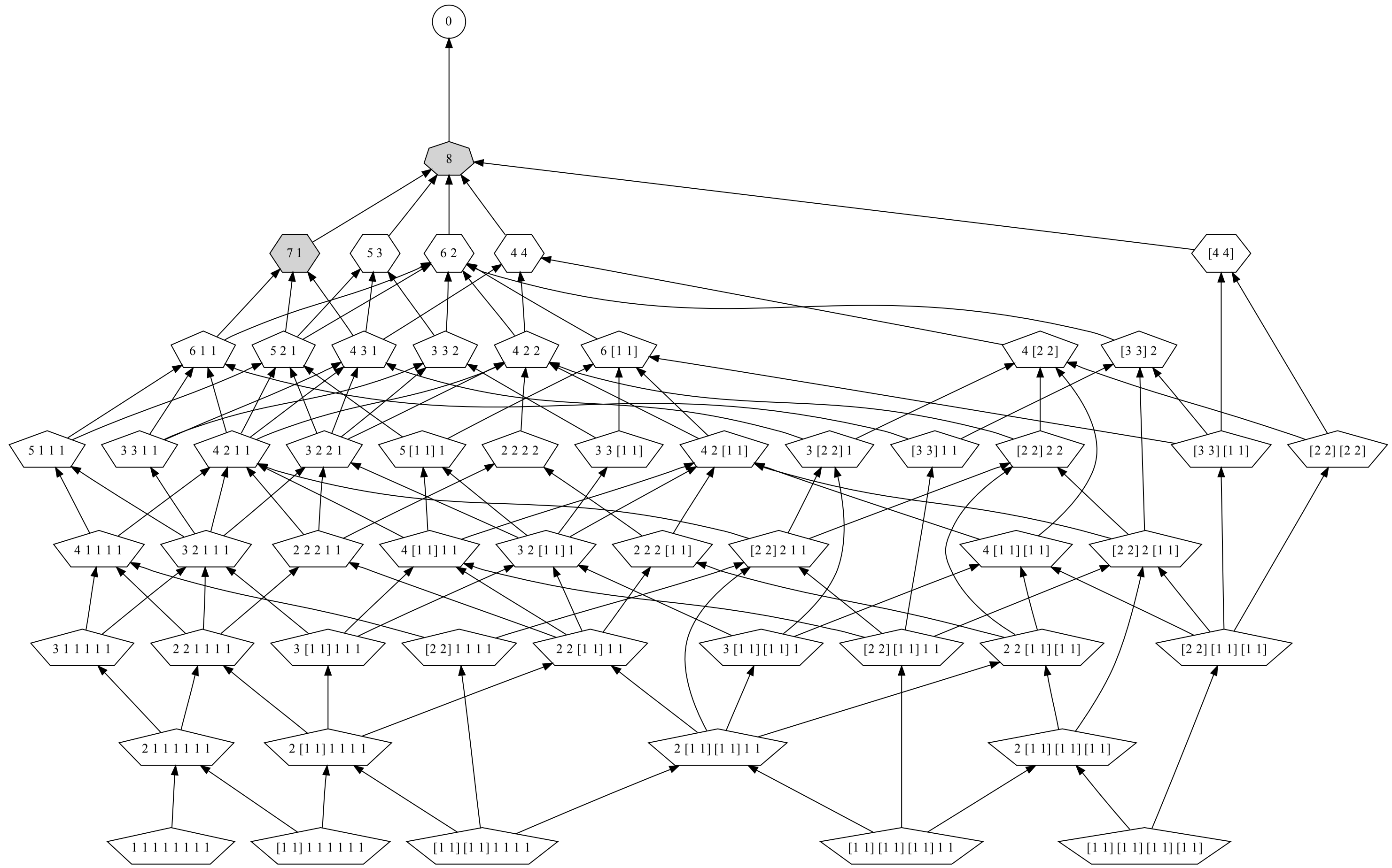

FIGURE 8.1: The leaf classification of 2,3-integrable $GL(2)$-structures of degree 4. Shaded nodes indicate classes that have been explicitly integrated in Section 8.3. The number of sides of the node is the total dimension of the reduced bundle over a 5-manifold.

9

PDEs and GL(2)-Structures in Degree Four

Now that a classification exists for 2,3-integrable $GL(2)$ structures, new questions present themselves regarding the motivating problem. Can we determine the torsion of the $GL(2)$-structures for the PDEs studied by [FHK07]? More broadly, which $GL(2)$-equivalence classes can be obtained from a PDE as described in [FHK07]? Do any root-types correspond to integrable PDEs not studied by [FHK07]? These questions amount to finding immersions of the structures into the symplectic group such that the base manifold locally embeds into the Lagrangian Grassmannian.

9.1 Symplectic Structures

Fix a symplectic 2-form σ on $\mathbb{R}^6$. The symplectic group is a Lie group of dimension 21 defined by

$$Sp(3) = \{A \in GL(6,\mathbb{R}) \ : \ \sigma(Ax, Ay) = \sigma(x,y) \ \forall x, y \in \mathbb{R}^6\}. \tag{9.1}$$

and the Lagrangian Grassmannian is $\Lambda_\sigma = \{P \in Gr_3(\mathbb{R}^6) \ : \ \sigma|_P = 0\}$. The symplectic group $Sp(3)$ has Maurer–Cartan form

$$\eta = \begin{pmatrix} \beta & \gamma \\ \alpha & -\beta^t \end{pmatrix} = \begin{pmatrix} \beta^1_1 & \beta^1_2 & \beta^1_3 & \gamma^{11} & \gamma^{12} & \gamma^{13} \\ \beta^2_1 & \beta^2_2 & \beta^2_3 & \gamma^{12} & \gamma^{22} & \gamma^{23} \\ \beta^3_1 & \beta^3_2 & \beta^3_3 & \gamma^{13} & \gamma^{23} & \gamma^{33} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} & -\beta^1_1 & -\beta^2_1 & -\beta^3_1 \\ \alpha_{12} & \alpha_{22} & \alpha_{13} & -\beta^1_2 & -\beta^2_2 & -\beta^3_2 \\ \alpha_{13} & \alpha_{13} & \alpha_{33} & -\beta^1_3 & -\beta^2_3 & -\beta^3_3 \end{pmatrix}, \alpha = \alpha^t, \gamma = \gamma^t. \tag{9.2}$$

The Maurer–Cartan formula for η is

$$0 = \mathrm{d}\eta + \eta \wedge \eta = \begin{pmatrix} \mathrm{d}\beta + \beta \wedge \beta + \gamma \wedge \alpha & \mathrm{d}\gamma + \beta \wedge \gamma - \gamma \wedge \beta^t \\ \mathrm{d}\alpha + \alpha \wedge \beta - \beta^t \wedge \alpha & -\mathrm{d}\beta^t + \alpha \wedge \gamma + \beta^t \wedge \beta^t \end{pmatrix}. \tag{9.3}$$

At the identity in $Sp(3)$, η restricts to the fiber as the condition $\alpha = 0$, so α is semi-basic for the bundle $Sp(3) \to \Lambda$.

If $B \to M$ is a $GL(2)$-structure arising from a second-order PDE as in Chapter 1, then M is a hyper-surface in Λ^o, and the distribution of rational-normal cones $\mathbf{C}$ over M is respected by the $Sp(3)$ action on Λ^o. That is, B maps into $Sp(3)$ such that M is a hyper-surface in Λ and such that the fibers of B (the symmetries of $\mathbf{C}$) are immersed into the fibers of $Sp(3)$.

To determine if such a map exists for a given B, it suffices to attempt to construct a $\mathfrak{sp}(3)$-valued 1-form on B such that the Maurer–Cartan formula is satisfied. At the risk of causing some minor confusion, we also refer to this section of $\mathbf{T}^*B \otimes \mathfrak{sp}(3)$ as η, so the requirement is that $\mathrm{d}\eta + \eta \wedge \eta = 0$ where $\alpha_{ij} = A_{ija}\omega^a$. If such a map η exists, then by Fundamental Lemma of Lie Groups, B admits a local map to $Sp(3)$ as desired [IL03, Theorem 1.6.10].

9.2 Solving Maurer–Cartan

Note that $\alpha : \mathcal{V}_4 \to \mathrm{Sym}^2(\mathbb{R}^3) = \mathrm{Sym}^2(\mathcal{V}_2)$, and recall that for $u, v \in \mathcal{V}_2$ the symmetric tensor is given by

$$\begin{pmatrix} u_{-2} \\ u_0 \\ v_2 \end{pmatrix} \circ \begin{pmatrix} v_{-2} \\ v_0 \\ v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_{-2}v_{-2} + v_{-2}u_{-2} & u_{-2}v_0 + v_{-2}u_0 & u_{-2}v_2 + v_{-2}u_2 \\ u_0v_{-2} + v_0u_{-2} & u_0v_0 + v_0u_0 & u_0v_2 + v_0u_2 \\ u_2v_{-2} + v_2u_{-2} & u_2v_0 + v_2u_0 & u_2v_2 + v_2u_2 \end{pmatrix} \tag{9.4}$$

In particular, the degrees in an element of $\mathrm{Sym}^2(\mathcal{V}_2)$ appear as follows:

$$\begin{pmatrix} -4 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix} \tag{9.5}$$

Therefore, α must have the following form, for constants A_{-4}, A_{-2}, A_0, A_0', A_2, and A_4:

$$\alpha = \begin{pmatrix} A_{-4}\ \omega^{-4} & A_{-2}\ \omega^{-2} & A_0\ \omega^0 \\ A_{-2}\ \omega^{-2} & A_0'\ \omega^0 & A_2\ \omega^2 \\ A_0\ \omega^0 & A_0\ \omega^2 & A_4\ \omega^4 \end{pmatrix}. \tag{9.6}$$

Writing $\beta = \beta_\varphi(\varphi) + \beta_\lambda(\lambda) + \beta_T(\omega)$, it is apparent that β_φ must be the representation $\mathfrak{sl}(2) \to M_{3\times 3}(\mathbb{R})$ such that the natural action of $M_{3\times 3}(\mathbb{R})$ on $\mathrm{Sym}^2(\mathcal{V}_2)$ is induced by the natural action of $\mathfrak{sl}(2)$ on $\omega \in \mathcal{V}_4$. One can now easily to verify the following formulas:

$$\begin{aligned} \alpha &= \begin{pmatrix} \omega^{-4} & \omega^{-2} & \omega^0 \\ \omega^{-2} & \omega^0 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{pmatrix}, \\ \beta &= \begin{pmatrix} 4\varphi_0 & 2\varphi_2 & 0 \\ -4\varphi_{-2} & 0 & 4\varphi_2 \\ 0 & -2\varphi_{-2} & -4\varphi_0 \end{pmatrix} - \frac{1}{2}\lambda I + \beta_T(\omega) \end{aligned} \tag{9.7}$$

All that remains is to find β_T and γ. This is "merely" arithmetic, though it presses the memory limits of common computer hardware (2 to 4 GiB) using systems such as Maple and Macaulay2 unless the programmer is careful to limit the number of variables in each stage of the process. The computation can be achieved as follows. Write β_T and γ as arbitrary linear combinations of ω and φ, though one must allow the coefficients to be functions, as they undoubtedly depend upon polynomials in T. In particular, set

$$\begin{cases} (\beta_T)^i_j = b^i_{j,a}\omega^a \\ (\gamma)^{i,j} = c^{i,j}_a\omega^a + \hat{c}^{i,j}_a\varphi^a \end{cases} \tag{9.8}$$

Using the dα part of Equation (9.3), we discover the following relations:

$$\begin{array}{lll}
b^1_{1,-4} = b^1_{1,-4} & b^1_{1,-2} = b^1_{1,-2} & b^1_{1,0} = b^1_{1,0} \\
b^1_{1,2} = 1290240T_{-2} & b^1_{1,4} = -322560T_{-4} & b^1_{2,-4} = -322560T_6 \\
b^1_{2,-2} = 1612800T_4 + b^1_{1,-4} & b^1_{2,0} = b^1_{1,-2} - 3225600T_2 & b^1_{2,2} = b^1_{1,0} + 3225600T_0 \\
b^1_{2,4} = -322560T_{-2} & b^1_{3,-4} = -322560T_8 & b^1_{3,-2} = 1290240T_6 \\
b^1_{3,0} = -1612800T_4 + b^1_{1,-4} & b^1_{3,2} = b^1_{1,-2} & b^1_{3,4} = b^1_{1,0} + 1612800T_0 \\
b^2_{1,-4} = b^1_{1,-2} - 645120T_2 & b^2_{1,-2} = b^2_{1,-2} & b^2_{1,0} = b^2_{1,0} \\
b^2_{1,2} = -2580480T_{-4} & b^2_{1,4} = 645120T_{-6} & b^2_{2,-4} = 645120T_4 \\
b^2_{2,-2} = -3870720T_2 + b^1_{1,-2} & b^2_{2,0} = b^2_{1,-2} + 6451200T_0 & b^2_{2,2} = -6451200T_{-2} + b^2_{1,0} \\
b^2_{2,4} = 645120T_{-4} & b^2_{3,-4} = 645120T_6 & b^2_{3,-2} = -2580480T_4 \\
b^2_{3,0} = 2580480T_2 + b^1_{1,-2} & b^2_{3,2} = b^2_{1,-2} & b^2_{3,4} = -3225600T_{-2} + b^2_{1,0} \\
b^3_{1,-4} = b^1_{1,0} + 1612800T_0 & b^3_{1,-2} = -2580480T_{-2} + b^2_{1,0} & b^3_{1,0} = b^3_{1,0} \\
b^3_{1,2} = 1290240T_{-6} & b^3_{1,4} = -322560T_{-8} & b^3_{2,-4} = -322560T_2 \\
b^3_{2,-2} = b^1_{1,0} + 3225600T_0 & b^3_{2,0} = -5806080T_{-2} + b^2_{1,0} & b^3_{2,2} = 3225600T_{-4} + b^3_{1,0} \\
b^3_{2,4} = -322560T_{-6} & b^3_{3,-4} = -322560T_4 & b^3_{3,-2} = 1290240T_2 \\
b^3_{3,0} = b^1_{1,0} & b^3_{3,2} = -2580480T_{-2} + b^2_{1,0} & b^3_{3,4} = 1612800T_{-4} + b^3_{1,0}
\end{array}$$

To simplify matters we make the assumption that any $b^i_{j,a}$ that is not explicitly solved for T is identically zero. That is, $b^i_{j,a} = 0$ unless it appears in the following list:

$$
\begin{array}{lll}
b^1_{1,2} = 1290240T_{-2} & b^1_{1,4} = -322560T_{-4} & b^1_{2,-4} = -322560T_6 \\
b^1_{2,-2} = 1612800T_4 & b^1_{2,0} = 3225600T_2 & b^1_{2,2} = 3225600T_0 \\
b^1_{2,4} = -322560T_{-2} & b^1_{3,-4} = -322560T_8 & b^1_{3,-2} = 1290240T_6 \\
b^1_{3,0} = -1612800T_4 & b^1_{3,4} = 1612800T_0 & b^2_{1,-4} = 645120T_2 \\
b^2_{1,2} = -2580480T_{-4} & b^2_{1,4} = 645120T_{-6} & b^2_{2,-4} = 645120T_4 \\
b^2_{2,-2} = -3870720T_2 & b^2_{2,0} = 6451200T_0 & b^2_{2,2} = -6451200T_{-2} \\
b^2_{2,4} = 645120T_{-4} & b^2_{3,-4} = 645120T_6 & b^2_{3,-2} = -2580480T_4 \\
b^2_{3,0} = 2580480T_2 & b^2_{3,4} = -3225600T_{-2} & b^3_{1,-4} = 1612800T_0 \\
b^3_{1,-2} = -2580480T_{-2} & b^3_{1,2} = 1290240T_{-6} & b^3_{1,4} = -322560T_{-8} \\
b^3_{2,-4} = -322560T_2 & b^3_{2,-2} = 3225600T_0 & b^3_{2,0} = -5806080T_{-2} \\
b^3_{2,2} = 3225600T_{-4} & b^3_{2,4} = -322560T_{-6} & b^3_{3,-4} = -322560T_4 \\
b^3_{3,-2} = 1290240T_2 & b^3_{3,2} = -2580480T_{-2} & b^3_{3,4} = 1612800T_{-4}
\end{array}
$$

Hence, the coefficients of β_T can be found explicitly in terms of the torsion T. Now, using the $\mathrm{d}\beta$ part of equation 9.3, all of the coefficients of γ can be established:

$$\hat{c}_{-2}^{1,1} = 1290240T_6 \qquad \hat{c}_0^{1,1} = 0 \qquad \hat{c}_2^{1,1} = 1290240T_2$$

$$\hat{c}_{-2}^{1,2} = -6451200T_4 \qquad \hat{c}_0^{1,2} = 0 \qquad \hat{c}_2^{1,2} = 0$$

$$\hat{c}_{-2}^{1,3} = 12902400T_2 \qquad \hat{c}_0^{1,3} = 0 \qquad \hat{c}_2^{1,3} = -7741440T_{-2}$$

$$\hat{c}_{-2}^{2,2} = 10321920T_2 \qquad \hat{c}_0^{2,2} = 0 \qquad \hat{c}_2^{2,2} = 10321920T_{-2}$$

$$\hat{c}_{-2}^{2,3} = -25804800T_0 \qquad \hat{c}_0^{2,3} = 0 \qquad \hat{c}_2^{2,3} = 15482880T_{-4}$$

$$\hat{c}_{-2}^{3,3} = 7741440T_{-2} \qquad \hat{c}_0^{3,3} = 0 \qquad \hat{c}_2^{3,3} = -7741440T_{-6}$$

$$\begin{aligned}
c_{-4}^{1,1} =& -1040449536000T_0T_8 + 4161798144000T_2T_6 - 2601123840000T_4^2 \\
c_{-2}^{1,1} =& 2080899072000T_2T_4 - 7283146752000T_0T_6 + 3537528422400T_{-2}T_8 \\
c_0^{1,1} =& 5202247680000T_0T_4 + 416179814400T_{-2}T_6 - 2705168793600T_{-4}T_8 \\
& - 2080899072000T_2^2 \\
c_2^{1,1} =& 1040449536000T_0T_2 + 1040449536000T_{-6}T_8 + 416179814400T_{-4}T_6 \\
c_4^{1,1} =& -208089907200T_{-2}T_2 - 148635648000T_{-6}T_6 - 163499212800T_{-8}T_8 \\
& - 2601123840000T_0^2
\end{aligned}$$

$$\begin{aligned}
c_{-4}^{1,2} =& -1040449536000T_2T_4 - 2080899072000T_0T_6 + 2080899072000T_{-2}T_8 \\
c_{-2}^{1,2} =& 14566293504000T_0T_4 - 8323596288000T_{-2}T_6 - 2913258700800T_{-4}T_8 \\
c_0^{1,2} =& -14566293504000T_0T_2 + 1248539443200T_{-6}T_8 + 11653034803200T_{-4}T_6 \\
c_2^{1,2} =& -4161798144000T_{-4}T_4 + 5202247680000T_0^2 - 4994157772800T_{-6}T_6 \\
& - 208089907200T_{-8}T_8 \\
c_4^{1,2} =& 832359628800T_{-8}T_6 + 1040449536000T_{-6}T_4 + 4161798144000T_{-2}T_0
\end{aligned}$$

$$
\begin{aligned}
c_{-4}^{1,3} =& -12485394432 00T_{-4}T_8 + 20808990720000T_2^2 - 3745618329600T_{-2}T_6\\
& + 2601123840000T_0T_4\\
c_{-2}^{1,3} =& -24970788864000T_0T_2 + 18311911833600T_{-2}T_4 + 832359628800T_{-6}T_8\\
& + 8323596288000T_{-4}T_6\\
c_0^{1,3} =& -13317754060800T_{-2}T_2 - 4518523699200T_{-6}T_6 - 163499212800T_{-8}T_8\\
& + 33814609920000T_0^2 - 23306069606400T_{-4}T_4\\
c_2^{1,3} =& 14150113689600T_{-4}T_2 + 832359628800T_{-8}T_6 + 10404495360000T_{-6}T_4\\
& - 16647192576000T_{-2}T_0\\
c_4^{1,3} =& -1768764211200T_{-8}T_4 - 2913258700800T_{-6}T_2 + 1248539443200T_{-2}^2
\end{aligned}
$$

$$
\begin{aligned}
c_{-4}^{2,2} =& -3329438515200T_{-4}T_8 + 3329438515200T_{-2}T_6 + 1664719257600T_2^2\\
c_{-2}^{2,2} =& 18311911833600T_{-4}T_6 - 18311911833600T_{-2}T_4 - 8323596288000T_0T_2\\
& + 1664719257600T_{-6}T_8\\
c_0^{2,2} =& -34959104409600T_{-4}T_4 - 10404495360000T_0^2 - 9274864435200T_{-6}T_6\\
& - 297271296000T_{-8}T_8 + 58265174016000T_{-2}T_2\\
c_2^{2,2} =& -4994157772800T_{-4}T_2 - 8323596288000T_{-2}T_0 + 1664719257600T_{-8}T_6\\
& + 18311911833600T_{-6}T_4\\
c_4^{2,2} =& -8323596288000T_{-2}^2 - 3329438515200T_{-8}T_4
\end{aligned}
$$

$$c^{2,3}_{-4} = 41617981440000T_{-4}T_6 - 41617981440000T_0T_2 + 16647192576000T_{-6}T_8$$

$$c^{2,3}_{-2} = -21641350348800T_{-4}T_4 + 26011238400000T_0^2 - 10701766656000T_{-6}T_6 - 3269984256000T_{-8}T_8$$

$$c^{2,3}_{0} = 2080899072000T_{-8}T_6 + 28300227379200T_{-6}T_4 - 60346073088000T_{-2}T_0 + 44947419955200T_{-4}T_2$$

$$c^{2,3}_{2} = -5410337587200T_{-8}T_4 - 38288542924800T_{-6}T_2 + 104044953600000T_{-4}T_0 + 16647192576000T_{-2}^2$$

$$c^{2,3}_{4} = -24970788864000T_{-4}T_{-2} + 7075056844800T_{-8}T_2 + 2080899072000T_{-6}T_0$$

$$c^{3,3}_{-4} = 12485394432000T_{-2}T_2 - 3121348608000T_{-8}T_8 - 2080899072000T_{-6}T_6 - 26011238400000T_0^2$$

$$c^{3,3}_{-2} = 10820675174400T_{-6}T_4 + 2080899072000T_{-8}T_6 + 2080899072000T_{-2}T_0$$

$$c^{3,3}_{0} = -5826517401600T_{-8}T_4 + 5826517401600T_{-2}^2 - 22473709977600T_{-6}T_2$$

$$c^{3,3}_{2} = -124853944320000T_{-4}T_{-2} + 8739776102400T_{-8}T_2 + 18728091648000T_{-6}T_0$$

$$c^{3,3}_{4} = 24970788864000T_{-6}T_{-2} - 62426972160000T_{-8}T_0$$

All of the terms of η are now fixed, and one must merely verify that the $\mathrm{d}\gamma$ block of Equation (9.3) is satisfied. (It is!) Because Equation (9.3) is satisfied identically for any choice of T, there is a local bundle embedding $B \to Sp(3)$ for any of the 55 root-types. Since a Maurer–Cartan-valued 1-form η can be found for an open set of $T \in \mathcal{V}_8$, this verifies the result of [FHK07] that there is an open orbit of PDEs with a three-dimensional family of hydrodynamic reductions. Moreover, this particular 1-form η is well-defined for *any* value of T. Hence, a local embedding exists for any representative of any of the 55 equivalence classes. This provides a perfect converse to Theorem 1.7.

Theorem 9.1 (All types are PDEs)**.** *Let* (B, M, p) *be a connected pointed 2,3-integrable* $GL(2)$*-structure over* M *of dimension 5. There is a neighborhood* $N \subset M$ *of* p *such that* N *embeds into* Λ^o *and the fibers of* $B|_N$ *immerse into the fibers of* $Sp(3)$*. In particular,* N *is an open subset of* $\{U \in \Lambda^o : F(U) = 0\}$ *for some PDE* $F(U) = 0$ *that is solvable by means of 3-parameter hydrodynamic reductions.*

10

Equivalence in Arbitrary Degree

In this chapter we solve the equivalence problem for arbitrary degree $GL(2)$-structures. The result is essentially identical to the degree 4 case examined in Theorem 4.1. The resulting connection is used in Chapters 11 and 12 to study 2- and 3-integrability in all higher degrees.

Let M be a manifold of dimension $n+1 \geq 5$, and let B be a $GL(2)$-structure over M. Then B admits structure equations

$$\mathrm{d}\omega = -\langle \varphi, \omega \rangle_1 - \langle \lambda, \omega \rangle_0 + T(\omega \wedge \omega) \tag{10.1}$$

where $\omega \in \Gamma(\mathbf{T}^* B \otimes \mathcal{V}_n)$ is the tautological form, and $(\varphi, \lambda) \in \Gamma(\mathbf{T}^* B \otimes [\mathfrak{sl}(2) \oplus \mathcal{V}_0])$ is a connection. Note that the torsion T lies in the space

$$\mathcal{V}_n \otimes (\wedge^2 \mathcal{V}_n^*) = \mathcal{V}_n \otimes (\mathcal{V}_{2n-2} \oplus \mathcal{V}_{2n-6} \oplus \cdots \oplus \mathcal{V}_{2n-2\nu}) . \tag{10.2}$$

$\nu = 2\lceil n/2 \rceil - 1$ is simply the largest odd number p for which the pairing $\langle \omega, \omega \rangle_p$ is nontrivial; it equals n if n is odd or $n-1$ if n is even.

How much torsion can be absorbed by a change of connection?

Theorem 10.1. *The map $\delta : \mathfrak{gl}(2) \otimes \mathcal{V}_n \to \mathcal{V}_n \otimes (\wedge^2 \mathcal{V}_n)$ has maximum rank. In other* *words, a change-of-connection $P, Q \in (\mathcal{V}_2 \oplus \mathcal{V}_0) \otimes \mathcal{V}_n^*$ allows absorption of torsion of weight $\mathcal{V}_{n-2}$, $\mathcal{V}_n$ (twice), and $\mathcal{V}_{n+2}$.*

The Clebsch–Gordon decomposition in Equation (10.2) breaks the torsion operator into irreducible components such that the torsion two-form $T(\omega \wedge \omega)$ is described by terms such as $\left\langle T_t^{2n-2p}, \langle \omega, \omega \rangle_p \right\rangle_q$ where $T_t^{2n-2p} \in \mathcal{V}_t$ and $n = t + (2n-2p) - 2q$.

For a connection $(\varphi, \lambda) \in \Gamma(\mathbf{T}^* B \otimes \mathfrak{gl}(2))$, a change of connection is given by ${}^*\varphi = \varphi + \delta P$ and ${}^*\lambda = \lambda + \delta Q$ where $(P, Q) \in \mathfrak{gl}(2) \otimes \mathcal{V}_n$ decomposes as

$$\begin{gathered} Q \in \mathcal{V}_n = \mathbb{R} \otimes \mathcal{V}_n, \quad \text{and} \\ P_{n-2} + P_n + P_{n+2} \in (\mathcal{V}_{n-2} \oplus \mathcal{V}_n \oplus \mathcal{V}_{n+2}) = \mathcal{V}_2 \otimes \mathcal{V}_n = \mathfrak{sl}(2) \otimes \mathcal{V}_n. \end{gathered} \tag{10.3}$$

In binary-polynomial notation, the irreducible components of P and Q are written:

$$\begin{aligned} Q &= \sum_{i=0}^{n} Q_{2i-n} \binom{n}{i} x^{n-i} y^i \in \mathcal{V}_n, \\ P_n &= \sum_{j=0}^{n} P_{n,2j-n} \binom{n}{j} x^{n-j} y^j \in \mathcal{V}_n, \\ P_{n-2} &= \sum_{k=0}^{n-2} P_{n-2,2j-n+2} \binom{n-2}{j} x^{n-2-j} y^k \in \mathcal{V}_{n-2}, \\ P_{n+2} &= \sum_{l=0}^{n+2} P_{n+2,2l-n-2} \binom{n+2}{l} x^{n+2-l} y^l \in \mathcal{V}_{n+2}. \end{aligned} \tag{10.4}$$

Since δP and δQ represent the irreducible representations of torsion that can be absorbed by P and Q, Theorem 10.1 amounts to the claim that δQ, δP_n, δP_{n-2}, and δP_{n+2} are nontrivial and that δP_n is distinct from δQ. Hence, to prove Theorem 10.1 we explicitly compute the linear map δ on each component and verify that it has maximal rank.

Lemma 10.2. *δQ is non-trivial, so it has maximum rank n.*

Proof. The image of δQ is defined by

$$0 = \delta Q(\omega \wedge \omega) - Q(\omega) \wedge \omega. \tag{10.5}$$

By the definition of the action by pairing,

$$\mathcal{V} \ni Q(\omega) = \langle Q, \omega \rangle_n = n! \sum_{a=0}^{n} (-1)^a \binom{n}{a} Q_{n-2a}\ \omega^{2a-n}. \tag{10.6}$$

Therefore,

$$\begin{aligned} Q(\omega) \wedge \omega &= \langle \langle Q, \omega \rangle_n, \omega \rangle_0 = \langle Q, \omega \rangle_n \wedge \omega \\ &= \left(n! \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{n-2r}\ \omega^{2r-n} \right) \wedge \left(\sum_{s=0} \binom{n}{s} x^{n-s} y^s\ \omega^{2s-n} \right) \\ &= n! \sum_{r,s=0}^{n} (-1)^r \binom{n}{r} \binom{n}{s} Q_{n-2r} x^{n-s} y^s\ (\omega^{2r-n} \wedge \omega^{2s-n}) \end{aligned} \tag{10.7}$$

δQ may lie in any representations of weight n in the space of torsion operators, $\mathcal{V}_n \otimes (\wedge^2 \mathcal{V}_n)$. For $p \in \{1, 3, 5, \ldots, \nu\}$ there are coefficients $C_p(Q)$ such that

$$\delta Q(\omega \wedge \omega) = \sum_p C_p(Q) \left\langle Q, \langle \omega, \omega \rangle_p \right\rangle_{n-p}. \tag{10.8}$$

So, to compute the image of Q via δ is to compute the coefficients $C_p(Q)$. These we now compute:

$$\begin{aligned} \delta Q(\omega \wedge \omega) &= \sum_p C_p \left\langle Q, \langle \omega, \omega \rangle_p \right\rangle_{n-p} \\ &= \sum_p C_p \sum_{b=0}^{n-p} \frac{(-1)^b}{(n-p)!} \binom{n-p}{b} \frac{\partial^{n-p} Q}{\partial x^{n-p-b} \partial y^b} \frac{\partial^{n-p} \langle \omega, \omega \rangle_p}{\partial x^b \partial y^{n-p-b}} \\ &= \sum_p C_p \sum_{b=0}^{n-p} \sum_{a,i,r,s=0}^{n} Z Z' Q_{2i-n}\ x^{2n-i-r-s} y^{i+r+s-n}\ \omega^{2r-n} \wedge \omega^{2s-n}, \end{aligned} \tag{10.9}$$

where

$$Z = \binom{n}{p}\binom{n-p}{b}\binom{p}{a}\frac{(-1)^{a+b}n!n!(2n-r-s-p)!(r+s-p)!}{(p+b-i)!(i-b)!(2n-r-s-p-b)!(r+s+b-n)!},$$

$$Z' = \frac{1}{(n-r-p+a)!(r-a)!(n-s-a)!(s-p+a)!}. \tag{10.10}$$

Equation 10.5 must be solved simultaneously as an equation of $\mathcal{V}_n$-valued two-forms; there are $\binom{n+1}{2}$ vector equations to be simultaneously solved. If they are solvable for C_p, these equations must be essentially independent of r, s, a, and b. Of course, the map δ exists and is well-defined, so solutions $\{C_p\}$ must exist. In particular, we may compute all C_p by selecting convenient values of r and s. To this end, we now compute the $\omega^{-n} \wedge \omega^n$ component of Equation 10.5.

First consider the $\omega^{-n} \wedge \omega^n$ component of $\delta Q(\omega \wedge \omega)$. Such a term can arise if $(r, s) = (0, n)$ or $(r, s) = (n, 0)$. In the first case, an examination of Z' shows that $a = 0$, and in the second case, an examination of Z' shows that $a = p$. Hence, the $\omega^{-n} \wedge \omega^n$ component of $\delta Q(\omega \wedge \omega)$ simplifies to

$$2\sum_p C_p \sum_{b=0}^{n-p}\sum_{i=0}^{n} \binom{n}{p}^2 \binom{n-p}{b}^2 \binom{p}{a}\binom{p}{i-b} n!(-1)^b Q_{2i-n} x^{n-i} y^i \tag{10.11}$$

The $\omega^{-n} \wedge \omega^n$ term of $Q(\omega) \wedge \omega$ is $n!\,(Q_n y^n - (-1)^n Q_{-n} x^n)$. Therefore, $\{C_p(Q)\}$ may be computed by solving the vector equation:

$$(-1)^{n+1} n! \begin{pmatrix} Q_{-n} \\ 0 \\ \vdots \\ 0 \\ \pm Q_n \end{pmatrix} = A(Q) \begin{pmatrix} Q_{-n} \\ Q_{-n+2} \\ \vdots \\ Q_{n-2} \\ Q_n \end{pmatrix} \tag{10.12}$$

where $A_{j,i}(Q) = 2\sum_p C_p(Q) \sum_{b=0}^{n-p} \binom{n}{p}^2 \binom{n-p}{b}^2 \binom{p}{a}\binom{p}{i-b} n!(-1)^b \delta_{j,i}$. This is a set of (necessarily consistent) affine equations for $\{C_p(Q)\}$. The equation $A_{1,1}(Q) = \pm n!$

shows that the solution values of $\{C_p(Q)\}$ are non-zero. So, Q may be chosen to absorb a component of T of weight n. $\square$

Lemma 10.3. *δP_n is non-trivial, so it as maximum rank n. Furthermore, $\delta P_n \neq \delta Q$.*

Proof. $P_n(\omega) \in \mathcal{V}_2$, and the image of δP_n is defined by

$$0 = \delta P_n(\omega \wedge \omega) - P_n(\omega) \wedge \omega. \tag{10.13}$$

First, by the definition of the action by pairing,

$$\begin{aligned} P_n(\omega) &= \langle P_n, \omega \rangle_{n-1} \\ &= \sum_{a=0}^{n-1} \sum_{i,k=0}^{n} W \;\, P_{n,2i-n} \;\omega^{2k-n} \;\, x^{n-k-i+1} y^{k+i+1-n} \end{aligned} \tag{10.14}$$

where

$$W = \binom{n-1}{a} \frac{n(-1)^a n!}{(1+a-i)!(i-a)!(n-k-a)!(k+a+1-n)!}. \tag{10.15}$$

Notice that the summand is nontrivial only for $i \in \{a, a+1\}$ and $n-k \in \{a, a+1\}$. Hence,

$$\begin{aligned} P_n(\omega) &= \langle P_n, \omega \rangle_{n-1} \\ &= n(n!) \sum_{a=0}^{n-1} (-1)^a \binom{n-1}{a} \Big(P_{n,2a-n} \omega^{n-2a-2} x^2 + \\ &\quad \left(P_{n,2a-n} \omega^{n-2a} + P_{n,2a+2-n} \omega^{n-2a-2} \right) xy + P_{n,2a+2-n} \omega^{n-2a} y^2 \Big) \end{aligned} \tag{10.16}$$

Therefore we can compute the two-form

$$P_n(\omega) \wedge \omega = \left\langle \langle P_n, \omega \rangle_{n-1}, \omega \right\rangle_1 = \frac{\partial P_n(\omega)}{\partial x} \wedge \frac{\partial \omega}{\partial y} - \frac{\partial P_n(\omega)}{\partial y} \wedge \frac{\partial \omega}{\partial x}. \tag{10.17}$$

As in the previous lemma, we are concerned only with the $\omega^{-n} \wedge \omega^n$ component of $P_n(\omega) \wedge \omega$, which simplifies nicely to

$$n^2(n!) \left(P_{n,-n} x^n + 2P_{n,-n+2} x^{n-1} y - (-1)^n 2P_{n,n-2} x y^{n-1} - (-1)^n P_{n,n} y^n \right) \tag{10.18}$$

δP_n may lie in any representation of weight n in the space of torsion operators, $\mathcal{V}_n \otimes (\wedge^2 \mathcal{V}_n)$. For $p \in \{1, 3, 5, \ldots, \nu\}$ there are coefficients $C_p(P_n)$ such that

$$\delta P_n(\omega \wedge \omega) = \sum_p C_p(P_n) \left\langle P_n, \langle \omega, \omega \rangle_p \right\rangle_{n-p}. \tag{10.19}$$

This is exactly the same formula that appears for δQ, so when considering the $\omega^{-n} \wedge \omega^n$ component, the matrix $A_{i,j}(P_n)$ remains the same as in the proof of the previous Lemma. The $\{C_p(P_n)\}$ are determined by the equation

$$n^2 n! \begin{pmatrix} P_{n,-n} \\ 2P_{n,-n+2} \\ 0 \\ \vdots \\ 0 \\ (-1)^n 2P_{n,n-2} \\ (-1)^n P_{n,n} \end{pmatrix} = A(P_n) \begin{pmatrix} P_{n,-n} \\ P_{n,-n+2} \\ P_{n,-n+4} \\ \vdots \\ P_{n,n-4} \\ P_{n,n-2} \\ P_{n,n} \end{pmatrix} \tag{10.20}$$

The equation $A_{1,1}(P_n) = n!n^2$ shows that the solution values of $\{C_p(P_n)\}$ are non-zero and that they are not identical to the solution values of $\{C_p(Q)\}$. Hence, P_n may be chosen to absorb a component of T of weight n that is distinct from the component absorbed by Q. $\square$

Finally, we must verify that δP_{n-2} and δP_{n+2} are non-trivial (hence have maximum rank). These computations are analogous to those just demonstrated for Q and P_n, and the result is as desired.

Just as four canonical connections appeared in Theorem 4.1, Theorem 10.1 does not give a unique canonical connection, since we have not specified which representations of torsion of weights $n-2$, n (twice), and $n+2$ to absorb. However, $\mathfrak{gl}(2)^{(1)} = 0$ and in specific examples it appears the equations are such that one can always absorb torsion T_n^{2n-2p} where $2n - 2p$ is minimized.

11

Integrability in Degree Five

In this chapter we consider 2- and 3-integrability of $GL(2)$-structures of degree 5 using the connection found in Chapter 10. The methods are identical to those employed in Chapters 5 and 6, so many details are omitted. We discover that 2-integrable $GL(2)$-structures of degree 5 admit a local classification given by an unknown singular foliation of $\mathcal{V}_9$. Also, we discover that all 3-integrable $GL(2)$-structures of degree 5 are also 2-integrable.

In matrix form, the connection for a $GL(2)$-structure of degree 5 is

$$\begin{pmatrix} \lambda_0 - 10\phi_0 & 10\phi_{-2} & 0 & 0 & 0 & 0 \\ -2\phi_2 & \lambda_0 - 6\phi_0 & 8\phi_{-2} & 0 & 0 & 0 \\ 0 & -4\phi_2 & \lambda_0 - 2\phi_0 & 6\phi_{-2} & 0 & 0 \\ 0 & 0 & -6\phi_2 & \lambda_0 + 2\phi_0 & 4\phi_{-2} & 0 \\ 0 & 0 & 0 & -8\phi_2 & \lambda_0 + 6\phi_0 & 2\phi_{-2} \\ 0 & 0 & 0 & 0 & -10\phi_2 & \lambda_0 + 10\phi_0 \end{pmatrix} \tag{11.1}$$

Torsion for a degree 5 $GL(2)$-structure initially lies in $\mathcal{V}_5 \otimes (\mathcal{V}_8 \oplus \mathcal{V}_4 \oplus \mathcal{V}_0)$. After fixing the connection defined in Chapter 10, the torsion operator decomposes as

$$T = (T^4_1 + T^4_9) + (T^8_3 + T^8_5 + T^8_7 + T^8_9 + T^8_{11} + T^8_{13}). \tag{11.2}$$

11.1 Bi-secant Surfaces and 2-Integrability

We want to find the conditions on B that allows any bi-secant plane in $\mathbf{T}M$ to be extended to a bi-secant surface $\Sigma \subset M$. The tangent planes $\mathbf{T}_p\Sigma$ must intersect $\mathbf{C}_p$ in two lines for all $p \in \Sigma$, so $\mathbf{T}_p\Sigma$ is spanned by $(a(p)x+b(p)y)^5$ and $(A(p)x+B(p)y)^5$. Under a $GL(2)$ change of basis in $\mathbf{T}_pM$, we may assume the spanning vectors are x^5 and y^5.

The corresponding ideal is generated by $\{\omega^{-3}, \omega^{-1}, \omega^1, \omega^3\}$ with independence condition $\omega^{-5} \wedge \omega^5 \neq 0$. The tableau and torsion for this ideal are given by

$$\mathrm{d}\begin{pmatrix}\omega^{-3}\\ \omega^{-1}\\ \omega^1\\ \omega^3\end{pmatrix} \equiv \begin{pmatrix}-2\phi_2 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 2\phi_{-2}\end{pmatrix} \wedge \begin{pmatrix}\omega^{-5}\\ \omega^5\end{pmatrix} + \begin{pmatrix}\tau^{-3}\\ \tau^{-1}\\ \tau^1\\ \tau^3\end{pmatrix} \omega^{-5} \wedge \omega^5, \tag{11.3}$$

modulo $\omega^{-3}, \omega^{-1}, \omega^1, \omega^3$. Of course τ^{-3} and τ^3 may be absorbed, but τ^{-1} and τ^1 must vanish if bi-secant surfaces exist. Examining the formulas for these terms, the vanishing of τ^{-1} and τ^1 forces $0 = T^4_1 = T^8_3 = T^8_5 = T^8_7 = T^8_{11} = T^8_{13}$ and $T^4_9 = 2T^8_9$. The remaining torsion component, T^8_9, is free.

The tableau in Equation (11.3) is involutive; solutions exist and depend on two functions of one variable. The first structure equation of a 2-integrable $GL(2)$-structure of degree 5 is

$$\mathrm{d}\omega = -\left\langle \varphi, \omega \right\rangle_1 - \left\langle \lambda, \omega \right\rangle_0 + \left\langle T^8_9, \left\langle \omega, \omega \right\rangle_1 \right\rangle_6 + 2\left\langle T^8_9, \left\langle \omega, \omega \right\rangle_3 \right\rangle_4. \tag{11.4}$$

To obtain additional necessary conditions, we must examine the Bianchi identity, $\nabla(\theta) \wedge \omega = \nabla(T(\omega \wedge \omega))$. Let $\mathcal{W}_9$ denote the subspace of $\mathcal{V}_9 \oplus \mathcal{V}_9$ given as the graph of the function $v \mapsto 2v$. Then 2-integrability implies $T : B \to \mathcal{W}_9$, so

$$\begin{aligned} \nabla T : & B \to \mathcal{W}_9 \otimes \mathcal{V}_5, \\ Q : & B \to \mathrm{Sym}^2(\mathcal{W}_9) \cap (\mathcal{V}_5 \otimes (\wedge^3 \mathcal{V}_5)). \end{aligned} \tag{11.5}$$

Of course, $\mathcal{W}_9$ is also an irreducible representation of $SL(2)$, and it is isomorphic to $\mathcal{V}_9$. The terms of ∇T and $Q(T,T)$ can be written in terms of ∇T^8_9 and $Q(T^8_9,T^8_9)$, but the pairings are not the same as they would be if $T=T^8_9$. Explicitly,

$$\begin{aligned}
0 &= \mathrm{d}(\mathrm{d}\omega) \\
&= \mathrm{d}\left(-\langle\varphi,\omega\rangle_1 - \lambda\wedge\omega + \left\langle T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6 + 2\left\langle T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4\right) \\
&= -\langle\mathrm{d}\varphi,\omega\rangle_1 - \langle\varphi,\langle\varphi,\omega\rangle_1\rangle_1 - \mathrm{d}\lambda\wedge\omega - \lambda\wedge\langle\varphi,\omega\rangle_1 - \langle\varphi,\lambda\wedge\omega\rangle \\
&\quad + \left\langle\varphi, \left\langle T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6\right\rangle_1 + 2\left\langle\varphi, \left\langle T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4\right\rangle_1 \\
&\quad + \lambda\wedge\left\langle T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6 + 2\lambda\wedge\left\langle T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4 \\
&\quad + \left\langle \mathrm{d}T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6 + 2\left\langle \mathrm{d}T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4 \\
&\quad - 2\left\langle T^8_9, \langle\langle\varphi,\omega\rangle_1,\omega\rangle_1\right\rangle_6 - 2\left\langle T^8_9, \langle\lambda\wedge\omega,\omega\rangle_1\right\rangle_6 \\
&\quad - 4\left\langle T^8_9, \langle\langle\varphi,\omega\rangle_1,\omega\rangle_3\right\rangle_4 - 4\left\langle T^8_9, \langle\lambda\wedge\omega,\omega\rangle_3\right\rangle_4 \\
&\quad + 2\left\langle T^8_9, \left\langle\left\langle T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6, \omega\right\rangle_1\right\rangle_6 + 4\left\langle T^8_9, \left\langle\left\langle T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4, \omega\right\rangle_1\right\rangle_6 \\
&\quad + 4\left\langle T^8_9, \left\langle\left\langle T^8_9, \langle\omega,\omega\rangle_1\right\rangle_6, \omega\right\rangle_3\right\rangle_4 + 8\left\langle T^8_9, \left\langle\left\langle T^8_9, \langle\omega,\omega\rangle_3\right\rangle_4, \omega\right\rangle_3\right\rangle_4 .
\end{aligned} \tag{11.6}$$

The last four terms provide $Q(T,T)(\omega\wedge\omega\wedge\omega)$, and the six terms before those provide $(\nabla T)(\omega\wedge\omega)$. By expanding this equation and applying Schur's lemma, the following relations are discovered:

$$S^8_{14} = \frac{105}{286}\left\langle T^8_9, T^8_9\right\rangle_2,\ \ S^8_{12} = 0 \tag{11.7}$$

$$R^8_{10} = \frac{70}{9}\left\langle T^8_9, T^8_9\right\rangle_4,\ \ S^8_{10} = \frac{10}{117}\left\langle T^8_9, T^8_9\right\rangle_4, \tag{11.8}$$

$$R^8_8 = 0,\ r^8_8 = 0,\ S^8_8 = 0, \tag{11.9}$$

$$R^8_6 = -\frac{56}{3}\left\langle T^8_9, T^8_9\right\rangle_6,\ \ R^4_6 = \frac{133}{2}\left\langle T^8_9, T^8_9\right\rangle_6,\ \ S^8_6 = \frac{7}{33}\left\langle T^8_9, T^8_9\right\rangle_6, \tag{11.10}$$

$$R^4_4 = 0,\ r^4_4 = 0,\ S^8_4 = 0, \tag{11.11}$$

$$R_2^4 = -784\left\langle T_9^8, T_9^8\right\rangle_8, \; R_2^0 = 16660\left\langle T_9^8, T_9^8\right\rangle_8, \tag{11.12}$$

$$r_0^0 = 0. \tag{11.13}$$

In particular, all terms in $\nabla\theta$ and ∇T depend only on the value of T. Hence, we have an existence and uniqueness theorem for 2-integrable $GL(2)$-structures of degree 5.

Theorem 11.1. *A $GL(2)$-structure B over M^6 is 2-integrable if only if there is an $SL(2)$-equivariant function $T : B \to \mathcal{V}_9$ such that the structure equations of B are of the form*

$$\begin{aligned}
\mathrm{d}\omega &= -\langle\varphi,\omega\rangle_1 - \langle\lambda,\omega\rangle_0 + \langle T, \langle\omega,\omega\rangle_1\rangle_6 + 2\langle T, \langle\omega,\omega\rangle_3\rangle_4\\
\mathrm{d}\lambda &= 0\\
\mathrm{d}\varphi &= -\frac{1}{2}\langle\varphi,\varphi\rangle_1 + \frac{70}{9}\left\langle\left\langle T_9^8, T_9^8\right\rangle_4, \langle\omega,\omega\rangle_1\right\rangle_8 - \frac{56}{3}\left\langle\left\langle T_9^8, T_9^8\right\rangle_6, \langle\omega,\omega\rangle_1\right\rangle_6\\
&\quad + \frac{133}{2}\left\langle\left\langle T_9^8, T_9^8\right\rangle_6, \langle\omega,\omega\rangle_3\right\rangle_4 - 784\left\langle\left\langle T_9^8, T_9^8\right\rangle_8, \langle\omega,\omega\rangle_3\right\rangle_2\\
&\quad + 16660\left\langle\left\langle T_9^8, T_9^8\right\rangle_8, \langle\omega,\omega\rangle_5\right\rangle_0\\
\mathrm{d}T &= K(T)\begin{pmatrix}\omega\\ \lambda\\ \varphi\end{pmatrix}
\end{aligned} \tag{11.14}$$

for a 10×10 matrix $K(T)$ whose entries are linear and quadratic polynomials in the coefficients of T. Moreover, if B is analytic, bi-secant surfaces locally depend upon two functions of one variable.

Just as in Chapter 6, Cartan's structure theorem applies, so the K-leaves of $\mathcal{V}_9$ determine leaf-equivalence classes of 2-integral $GL(2)$-structures of degree 5. Unfortunately, a simple computation (for MAPLE) shows that, though the discriminant of T and the determinant of $K(T)$ are both polynomials of degree 16, they have no common divisors, and there appears to be no obvious relation between the multiplicity of the roots of T and the rank of $K(T)$. At this point, there is little hope of explicitly identifying the leaf-equivalence classes by analyzing the K-leaves.

11.2 Tri-secant 3-Folds and 3-Integrability

Now let us find the conditions for extension of any tri-secant subspace of $\mathbf{T}M$ to a tri-secant submanifold N of M. Under a $GL(2)$ change of basis, we may assume the spanning vectors are x^5 and y^5 and $(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$. Hence, any vector in $\mathbf{T}N \subset \mathbf{T}M$ looks like $(a+b)\ x^5 + b\ 5x^4y + b\ 10x^3y^2 + b\ 10x^2y^3 + b\ 5xy^4 + (b+c)\ y^5$. Lifting this problem to B, these vectors are in the kernel of three 1-forms, $\kappa^{-3} = \omega^{-3} - \omega^1$, $\kappa^{-1} = \omega^{-1} - \omega^1$, and $\kappa^3 = \omega^3 - \omega^1$. Hence, we study the EDS differentially generated by these κ's with the independence condition $\omega^{-5} \wedge \omega^1 \wedge \omega^5 \neq 0$. The torsion and tableau for this system are given by:

$$\mathrm{d}\begin{pmatrix}\kappa^{-3}\\ \kappa^{-1}\\ \kappa^3\end{pmatrix} \equiv \begin{pmatrix} 2\varphi_2 & \lambda_0 + 2\varphi_0 & 0\\ 0 & \lambda_0 + 6\varphi_{-2} + 2\varphi_0 & 0\\ 0 & \lambda_0 + 2\varphi_0 + 8\varphi_2 & -2\varphi_{-2}\end{pmatrix} \wedge \begin{pmatrix}\omega^5\\ \omega^1\\ \omega^5\end{pmatrix} + \tau(\omega\wedge\omega), \tag{11.15}$$

modulo $\kappa^{-3}, \kappa^{-1}, \kappa^3$. Some components of τ are unabsorbable, and these force the conditions $0 = T^4_1 = T^8_3 = T^8_5 = T^8_7 = T^8_{11} = T^8_{13}$ and $T^4_9 = 2T^8_9$. The remaining torsion component, T^8_9, is free. Application of Cartan's test shows that the tableau is involutive, with solutions depending on three functions of one variable.

Theorem 11.2. *A $GL(2)$-structure of degree 5 is 3-integrable if and only if it is 2-integrable. If the structure is analytic, then tri-secant 3-folds exist and locally depend on three functions of one variable.*

Compared to the case of degree 4 in Chapter 6, it is striking that closed structure equations are obtained using only the condition of 2-integrability, and that the necessary and sufficient conditions for 2-integrability and 3-integrability are identical. No prolongation is required, unlike the case in Chapter 6. In Chapter 12, this behavior appears to hold for all higher degree $GL(2)$-structures as well.

12

Integrability for Large Degrees

This chapter is a summary of results regarding 2- and 3-integrability of $GL(2)$-structures of degree $n \geq 6$. The primary conjecture is that 2-integrable structures are characterized locally by a single irreducible component of torsion of degree $n+4$, and all 3-integrable structures are also 2-integrable. In other words, the situation in degree 5 is expected to extend to all degrees $n \geq 6$, too.

12.1 Bi-secant Surfaces and 2-Integrability

The vectors spanning the bi-secant plane can be written as x^n and y^n, so the existence of a bi-secant surface through every bi-secant tangent plane is dictated by the integrability of the ideal $\mathcal{I}$ that is differentially generated by ω^{-n+2}, ω^{-n+4}, $\ldots$, ω^{n-4}, and ω^{n-2} and with the independence condition $\omega^{-n} \wedge \omega^n \neq 0$. The tableau and torsion are given by

$$\mathrm{d}\begin{pmatrix} \omega^{-n+2} \\ \omega^{-n+4} \\ \vdots \\ \omega^{n-4} \\ \omega^{n-2} \end{pmatrix} \equiv \begin{pmatrix} \pi_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \pi_2 \end{pmatrix} \wedge \begin{pmatrix} \omega^{-n} \\ \omega^n \end{pmatrix} + \begin{pmatrix} \tau^{-n+2} \\ \tau^{-n+4} \\ \vdots \\ \tau^{n-4} \\ \tau^{n-2} \end{pmatrix} \omega^{-n} \wedge \omega^n, \tag{12.1}$$

modulo $\omega^{-n+2}, \omega^{-n+4}, \ldots, \omega^{n-4}, \omega^{n-2}$. Cursory examination of the tableau indicates a simple theorem.

Theorem 12.1 (Existence of solutions)**.** *If an analytic $GL(2)$-structure of degree $n \geq 3$ is 2-integrable, then bi-secant surfaces exist and depend on two functions of one variable.*

The more difficult problem is to determine the conditions on torsion and curvature that allow 2-integrability. In order to carry out the integrability results for $n = 4$ and $n = 5$, Maple code was produced that is generic enough to study 2- or 3-integrability for any particular n. Using the canonical connection described in Chapter 10, it generates the ideal representing k-integrability and applies Jeanne Clelland's implementation of the Cartan–Kähler theorem to find the unabsorbable torsion, $\tau^{-n+4}, \ldots, \tau^{n-4}$, of the tableau. It then determines the conditions on T by finding the irreducible components of T that appear in τ. Finally, the code examines the Bianchi identity to determine the relations on the irreducible components of curvature. This is exactly the algorithm that produces Theorems 5.2, 6.4, 11.1, and 11.2.

Theorem 12.2. *Let $3 \leq n \leq 20$. If a $GL(2)$-structure of degree n is 2-integrable, then its torsion takes values only in irreducible representations of degree $n+4$. Moreover, the torsion is entirely determined by the irreducible component that appears in $T(\omega \wedge \omega)$ as $\left\langle T^{2n-2}_{n+4}, \langle\omega,\omega\rangle_1 \right\rangle_{n+1}$. In particular, there are constants $A_3, A_5, \ldots, A_\nu$ such that $T(\omega \wedge \omega)$ equals*

$$\left\langle T^{2n-2}_{n+4}, \langle\omega,\omega\rangle_1 \right\rangle_{n+1} + A_3 \left\langle T^{2n-2}_{n+4}, \langle\omega,\omega\rangle_3 \right\rangle_{n-1} + \cdots + A_\nu \left\langle T^{2n-2}_{n+4}, \langle\omega,\omega\rangle_\nu \right\rangle_{n-\nu+2}. \quad (12.2)$$

Theorem 12.2 is slightly more complicated for $n \geq 5$ than it is for $n = 3$ or $n = 4$. In the two smaller degrees, there is only one irreducible component of torsion that takes values in a representation of degree $n+4$; however, for $n \geq 5$, there are many. As seen in Chapter 11, multiple components of torsion of degree $n+4$ are non-vanishing, but the image of T lies in an oblique subspace $\mathcal{W}_{n+4} \subset \mathcal{V}_{n+4} \oplus \cdots \oplus \mathcal{V}_{n+4}$. This subspace $\mathcal{W}_{n+4}$ projects non-trivially onto each of the copies of $\mathcal{V}_{n+4}$, but by analogy to the $n = 3$ and $n = 4$ case, we choose to describe the term T^{2n-2}_{n+4} as the free coordinate.

The upper limit of $n = 20$ is simply where MAPLE extinguishes the memory of a desktop computer (2 GiB) while computing the torsion of the tableau. These results are expected to hold for larger n as well.

Under the condition of Theorem 12.2, the Bianchi identity can be examined; using Schur's lemma, relations between R, r, ∇T, and Q can be determined. If $6 \leq n \leq 9$, the result is analogous to Theorem 11.1: all components of R, r, and ∇T depend only on $Q(T, T)$.

Theorem 12.3 (Sufficient Conditions and Classification)**.** *Let $5 \leq n \leq 9$. If a $GL(2)$-structure of degree n is 2-integrable, then its structure equations satisfy Theorem 2.12 such that the local structure is determined by the value of T^{2n-2}_{n+4} at a point. In particular, the leaf-equivalence classification of pointed, connected 2-integrable structures is described by leaves of a singular foliation of $\mathcal{V}_{n+4}$.*

The upper limit of $n = 9$ is also arbitrary, given by computational limitations in the decomposition of Q.

12.2 Tri-secant Surfaces and 3-Integrability

Now let us find the conditions for extension of any tri-secant subspace of $\mathbf{T}M$ to a tri-secant submanifold N of M. Under a $GL(2)$ change of basis, we may assume the spanning vectors are x^n and y^n and $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$. Hence, any vector in $\mathbf{T}N \subset \mathbf{T}M$ looks like $(a+b)\ x^n + b\ nx^{n-1}y + \cdots + b\ nxy^{n-1} + (b+c)\ y^n$.

Lifting this problem to B, these vectors are in the kernel of $n-2$ 1-forms, $\kappa^{-n+4} = \omega^{-n+4} - \omega^{-n+2}$, $\kappa^{-n+6} = \omega^{-n+6} - \omega^{-n+2}$, ..., and $\kappa^{n-2} = \omega^{n-2} - \omega^{-n+2}$. Hence, we study the EDS differentially generated by these κ's with the independence condition $\omega^{-n} \wedge \omega^{-n+2} \wedge \omega^{n} \neq 0$. The tableau for this system is easier to understand if we use a different basis for the span of ω^{-n}, ω^{-n+2}, and ω^{n}; let $\eta^1 = \omega^{-n}$, $\eta^2 = \frac{1}{2}(\omega^{-n+2} - \omega^{n})$, and $\eta^3 = \frac{1}{2}(\omega^{-n+2} + \omega^{n})$.

The torsion and tableau for this system are given by

$$\mathrm{d}\begin{pmatrix} \kappa^{-n+4} \\ \kappa^{-n+6} \\ \vdots \\ \kappa^{n-6} \\ \kappa^{n-4} \\ \kappa^{n-2} \end{pmatrix} \equiv \begin{pmatrix} \pi_1 & 0\pi_1 + 1\pi_2 & 0\pi_1 + 1\pi_2 \\ \pi_1 & 1\pi_1 + 2\pi_2 & 1\pi_1 + 2\pi_2 \\ \vdots & \vdots & \vdots \\ \pi_1 & (n-5)\pi_1 + (n-4)\pi_2 & (n-5)\pi_1 + (n-4)\pi_2 \\ \pi_1 & (n-4)\pi_1 + (n-3)\pi_2 & (n-4)\pi_1 + (n-3)\pi_2 \\ \pi_1 & \pi_3 & (n-3)\pi_1 + (n-2)\pi_2 \end{pmatrix} \wedge \begin{pmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} + \tau(\eta, \eta). \tag{12.3}$$

modulo κ^{-n+4}, ..., κ^{n-4}, and κ^{n-2}. Again, examination of the tableau indicates that it is involutive. This is in contrast to the case $n = 4$, where prolongation is required.

Theorem 12.4 (Existence of solutions)**.** *If an analytic $GL(2)$-structure of degree $n \geq 5$ is 3-integrable, then tri-secant surfaces exist and depend on three functions of one variable.*

Some components of τ are unabsorbable, though they are more difficult to see than in the 2-integrable case. So, 3-integrability forces conditions on the torsion T of the 3-integrable $GL(2)$-structure. In all cases that the MAPLE code can analyze, the conditions for 3-integrability are identical to those of 2-integrability, just as seen in Chapter 11 for $n = 5$. On account of Theorem 12.3, this provides necessary and sufficient conditions as well as a local equivalence theorem and a leaf-classification.

Theorem 12.5 (Conditions and Classification)**.** *Let $5 \leq n \leq 20$. A $GL(2)$-structure of degree n is 2-integrable if and only if it is 3-integrable.*

12.3 Conjectured Results

Theorems 12.2, 12.3, and 12.5 are expected to hold for $n \geq 21$ as well, but MAPLE's data-types are not sufficiently flexible to allow direct computation for generic n.

Conjecture 12.6 (Structure Classification)**.** *A $GL(2)$-structure B of degree $n \geq 5$ is 2-integrable if and only if it is 3-integrable. In this case, the local structure in a connected neighborhood of $b \in B$ is determined by $T^{2n-2}_{n+4}(b)$. Moreover, the leaf-equivalence class of pointed, connected structures are determined by a singular foliation of $\mathcal{V}_{n+4}$.*

Included here are some *partial* results the point towards a direct proof of this conjecture. The technique follows the outline suggested by the MAPLE output for $6 \leq n \leq 20$. First, we try to determine which components of torsion must vanish because $\tau^{4-n}, \ldots, \tau^{n-4}$ are unabsorbable for a 2-integrable structure.

Conjecture 12.7 (n+4 Conjecture)**.** *The vanishing of $\tau^{4-n}, \ldots, \tau^{n-4}$ forces the vanishing of all irreducible components of $T(\omega \wedge \omega)$ except those of weight n+4. Of the components of weight n+4, exactly one is free.*

A proof of Conjecture 12.7 must involve a computation to determine which representations of T appear in the equation $0 = \tau^{-n+2i}(\omega^{-4} \wedge \omega^4)$ for $i = 2, \ldots, n-2$. Note that $\tau^{-n+2i}(\omega^{-4} \wedge \omega^4)$ is the $x^{n-i}y^i$ component of $T(\omega \wedge \omega)$ modulo $\{\omega^{-n+2}, \ldots, \omega^{n-2}\}$. Write $T^{2n-2p}_t = \sum_{s=0}^{t} T^{2n-2p}_{t,2s-t} \binom{t}{s} x^{t-s}y^s$ and $\langle \omega, \omega \rangle_p \equiv \frac{2}{p!} \left(\frac{n!}{(n-p)!}\right)^2 x^{n-p}y^{n-p}\omega^{-n} \wedge \omega^n$. Set $q(p) = (t + n - 2p)/2$, and we may compute

$$\begin{aligned} T(\omega \wedge \omega) &\equiv \sum_p \left\langle T^{2n-2p}_t, \frac{2}{p!} \left(\frac{n!}{(n-p)!}\right)^2 x^{n-p}y^{n-p} \right\rangle_q, \qquad \operatorname{mod} \omega^{2-n}, \ldots, \omega^{n-2}, \\ &= \sum_p \sum_{a=0}^{q} \sum_{s=0}^{t} \binom{q}{a} \frac{(-1)^a 2(n!)^2 t!\, T^{2n-2p}_{t,2s-t}\, x^{t-s-q+n-p} y^{s-a+n-p-q}\, \omega^{-n} \wedge \omega^n}{p!q!(t-s-q+a)!(s-a)!(n-p-a)!(n-p-q+a)!}. \end{aligned} \tag{12.4}$$

This only must vanish for $2 \leq s - a + n - p - q \leq n - 2$. To better identify the conditions on T_t^{2n-2p}, write

$$A(t,p,s,a) = \binom{q}{a} \frac{(-1)^a 2(n!)^2 t!}{p!q!(t-s-q+a)!(s-a)!(n-p-a)!(n-p-q+a)!}, \quad \text{and}$$

$$Z_p^i(t) = \sum_p \{A(t,p,s,a) \ : \ i = s - a + n - p - q\}. \tag{12.5}$$

Recall that ν is the largest odd number less than n. We now have a matrix $Z(t)$ that has rows indexed by $p \in \{1, 3, \ldots, \nu\}$ and columns indexed by $i \in \{2, 3, \ldots, n-2\}$, and the condition for existence of bi-secant surfaces is

$$0 = \tau^{n-2i} = \sum_p Z_p^i(t) T_t^{2n-2p}, \quad \text{for } i = 2, \ldots, n-2. \tag{12.6}$$

In particular, Conjecture 12.7 may be rephrased in terms of the matrix $Z(t)$.

Conjecture 12.8 (n+4 Conjecture, Matrix Form)**.** *The matrix $Z(t)$ has trivial kernel for $t \neq n+4$, and the kernel of $Z(n+4)$ is one-dimensional such that the $p = 1$ coordinate is free.*

The matrix $Z(j)$ admits many symmetries, but so far no success has been achieved in determining its exact kernel for arbitrary t.

13

Conclusion and Future Work

In the context of $GL(2)$ geometry, this dissertation has achieved some significant additions to the known results. Chapter 10 provides a handful of canonical connections on $GL(2)$-structures of any degree. Using one of these connections, the following results are now known for k-integrable $GL(2)$-structures over M^{n+1}, at least in the real-analytic category.

- $n = 3$, $k = 2$: Bi-secant surfaces locally depend on two functions of one variable. Torsion takes values in $\mathcal{V}_7$, and there is curvature in $\mathcal{V}_4$ and $\mathcal{V}_2$ [Bry91].
- $n = 4$, $k = 2$: Bi-secant surfaces locally depend on two functions of one variable. Torsion takes values in $\mathcal{V}_8$, and there is curvature in $\mathcal{V}_0$.
- $n = 4$, $k = 3$: Tri-secant 3-folds locally depend on three functions of one variable. If the structure is also 2-integrable, then torsion takes values in $\mathcal{V}_8$, and there is no free curvature. In this case, a local classification of the structures is given by root-types in $\mathcal{V}_8$. All classes arise from hydrodynamic PDEs.
- $5 \leq n$, $k = 2$: Bi-secant surfaces locally depend on two functions of one

variable. Partial Conjecture: Torsion takes values in $\mathcal{V}_{n+4}$, and there is no free curvature. A local classification exists but remains unidentified.

- $5 \leq n,\ k = 3$: Tri-secant 3-folds locally depend on three functions of one variable. Partial Conjecture: 3-integrability is equivalent to 2-integrability.

From the perspective of PDE theory, the case $n = 4$ is particularly interesting. In [FHK07] it is shown that any second-order PDE, $F(u_{ij}) = 0$, integrable via 3-parameter hydrodynamic reductions gives rise to a natural 2,3-integrable $GL(2)$-structure of degree 4. This dissertation classifies all 2,3-integrable $GL(2)$-structures of degree 4 and verifies that all components of the classification are obtainable from second-order PDE. To make the greatest use of this classification, two further results are desired.

1. Given a PDE of this type, find the corresponding $M \subset \Lambda^o$ and compute the torsion of the corresponding $GL(2)$-structure. A good starting point is the quasilinear case, $0 = F(u_{ij}) = \sum_{ij} f_{ij}(\xi)u_{ij}$, which was classified quite recently in [BFT08]. The quasilinear classification must somehow fit into the broader classification presented in this dissertation. Additionally, structure equations for the point symmetries of specifics PDEs can be computed using techniques like those shown in [LR98], [LR00], and [COP05], and these techniques may be applicable in a more general way to analyze this whole family of PDEs.

2. Given $v \in \mathcal{V}_8$, produce a PDE of this type whose associated $GL(2)$-structure has torsion v. This can probably be accomplished by manipulating the integration computations from Section 8.3 to embed the local coordinates for M into the standard local coordinates on Λ^o such that the $GL(2)$ fiber immerses into $Sp(3, \mathbb{R})$ as in Chapter 9. New integrable PDEs will probably arise this way.

These two results should be pursued immediately to finally establish a clear coordinate-independent geometric meaning behind hydrodynamic reductions.

More generally, $F(u_{ij}) = 0$ is not invariant under arbitrary coordinate changes, but integrability phenomena should remain independent of coordinate choice. The relationship between $F = 0$ and the $GL(2)$-structure over $M \subset \Lambda^o$ appears to require that F only depends on second derivatives. An important open problem is the extension of the theory of k-integrable $GL(2)$-structures to PDEs that include lower derivatives, $0 = F(u, u_i, u_{ij})$.

Regarding the high degree cases studied in Chapter 11 and Chapter 12, the leaf-classification is elusive, but it certainly exists. Presumably these structures represent *something*, even if it is not a hydrodynamic PDE. Thus there remain two results to pursue: tie the structures to a physical system or identify the foliation that classifies the structures. Either one would help the other and undoubtedly lead to new discoveries. Note that a correspondence to reduced jet-graphs could be sought for PDEs $F(u_{ij}) = 0$ with $1 \leq i, j \leq N$. This could conceivably lead to $GL(2)$-structures of degree $n = N(N+1)/2 - 2$. If 2,3-integrability holds, then a local classification of the structures is given by a singular foliation of $\mathcal{V}_{n+4}$, and one would seek local embeddings of these structures into $Sp(N)$.

Even more broadly, one hopes to establish a relationship between the $GL(2)$ geometry of PDEs and the $GL(2)$ geometry of ODEs presented in [Nur07], [BN07], and elsewhere. At a conference at the Mathematical Sciences Research Institute in May 2008, Nurowski indicated that the interesting $GL(2)$ geometry for ODEs appears to be limited to dimension 5 in much the same way that 3-integrable $GL(2)$-structures over M^{n+1} are most interesting for $n = 4$. The geometry is tantalizingly similar even though the irreducible torsion takes values in different representations.

Finally, it would be nice to put the 2-integrability structure classification (Conjecture 12.6) to rest once and for all. Simple-minded attempts of exploiting the symmetries in the matrix $Z(t)$ have so far failed, as have initial attempts at using the methods of hyper-geometric series as summarized in [PWZ96]. However, the examples studied by hand make it clear that this should require much determination but little cleverness for any given degree n.

Appendix A

The Matrix J

For reference, here is the matrix $J(T)$, listed by column.

The ω_{-4} component:

$$J_{1,1} = 2580480T_{-8}T_4 - 2580480T_{-6}T_2$$

$$J_{2,1} = -2257920T_{-4}T_2 + 645120T_{-8}T_6 + 1612800T_{-6}T_4$$

$$J_{3,1} = 645120T_{-4}T_4 - 1935360T_{-2}T_2 + 1198080T_{-6}T_6 + 92160T_{-8}T_8$$

$$J_{4,1} = -1612800T_0T_2 - 322560T_{-2}T_4 + 322560T_{-6}T_8 + 1612800T_{-4}T_6$$

$$J_{5,1} = 774144T_{-4}T_8 + 1806336T_{-2}T_6 - 1290240T_2^2 - 1290240T_0T_4$$

$$J_{6,1} = -3225600T_2T_4 + 1612800T_0T_6 + 1612800T_{-2}T_8$$

$$J_{7,1} = 3225600T_0T_8 - 3225600T_4^2$$

$$J_{8,1} = 6451200T_2T_8 - 6451200T_6T_4$$

$$J_{9,1} = -12902400T_6^2 + 12902400T_8T_4$$

The ω_{-2} component:

$$J_{1,2} = -12902400T_{-8}T_2 + 12902400T_{-6}T_0$$

$$J_{2,2} = -3548160T_{-8}T_4 - 7741440T_{-6}T_2 + 11289600T_{-4}T_0$$

$$J_{3,2} = -2580480T_{-4}T_2 + 9676800T_{-2}T_0 - 645120T_{-8}T_6 - 6451200T_{-6}T_4$$

$$J_{4,2} = -8386560T_{-4}T_4 + 2580480T_{-2}T_2 + 8064000T_0^2 - 2211840T_{-6}T_6 - 46080T_{-8}T_8$$

$$J_{5,2} = 14192640T_0T_2 - 8773632T_{-2}T_4 - 258048T_{-6}T_8 - 5160960T_{-4}T_6$$

$$J_{6,2} = -967680T_{-4}T_8 - 10321920T_{-2}T_6 + 12902400T_2^2 - 1612800T_0T_4$$

$$J_{7,2} = 19353600T_2T_4 - 16128000T_0T_6 - 3225600T_{-2}T_8$$

$$J_{8,2} = 22579200T_4^2 - 9676800T_0T_8 - 12902400T_2T_6$$

$$J_{9,2} = -25804800T_2T_8 + 25804800T_6T_4$$

The ω_0 component:

$$J_{1,3} = -25804800T_{-6}T_{-2} + 25804800T_{-8}T_0$$

$$J_{2,3} = -22579200T_{-4}T_{-2} + 8064000T_{-8}T_2 + 14515200T_{-6}T_0$$

$$J_{3,3} = 1935360T_{-8}T_4 - 19353600T_{-2}^2 + 14192640T_{-6}T_2 + 3225600T_{-4}T_0$$

$$J_{4,3} = 17418240T_{-4}T_2 - 24192000T_{-2}T_0 + 322560T_{-8}T_6 + 6451200T_{-6}T_4$$

$$J_{5,3} = 14450688T_{-4}T_4 + 3096576T_{-2}T_2 - 19353600T_0^2 + 1769472T_{-6}T_6 + 36864T_{-8}T_8$$

$$J_{6,3} = -24192000T_0T_2 + 17418240T_{-2}T_4 + 322560T_{-6}T_8 + 6451200T_{-4}T_6$$

$$J_{7,3} = 1935360T_{-4}T_8 + 14192640T_{-2}T_6 - 19353600T_2^2 + 3225600T_0T_4$$

$$J_{8,3} = -22579200T_2T_4 + 14515200T_0T_6 + 8064000T_{-2}T_8$$

$$J_{9,3} = 25804800T_0T_8 - 25804800T_2T_6$$

The ω_{-2} component:

$$J_{1,4} = 25804800T_{-4}T_{-6} - 25804800T_{-2}T_{-8}$$

$$J_{2,4} = -12902400T_{-6}T_{-2} - 9676800T_{-8}T_0 + 22579200T_{-4}^2$$

$$J_{3,4} = 19353600T_{-4}T_{-2} - 3225600T_{-8}T_2 - 16128000T_{-6}T_0$$

$$J_{4,4} = -967680T_{-8}T_4 + 12902400T_{-2}^2 - 10321920T_{-6}T_2 - 1612800T_{-4}T_0$$

$$J_{5,4} = -8773632T_{-4}T_2 + 14192640T_{-2}T_0 - 258048T_{-8}T_6 - 5160960T_{-6}T_4$$

$$J_{6,4} = -8386560T_{-4}T_4 + 2580480T_{-2}T_2 + 8064000T_0^2 - 2211840T_{-6}T_6 - 46080T_{-8}T_8$$

$$J_{7,4} = 9676800T_0T_2 - 2580480T_{-2}T_4 - 645120T_{-6}T_8 - 6451200T_{-4}T_6$$

$$J_{8,4} = 11289600T_0T_4 - 3548160T_{-4}T_8 - 7741440T_{-2}T_6$$

$$J_{9,4} = 12902400T_0T_6 - 12902400T_{-2}T_8$$

The ω_4 component:

$$J_{1,5} = 12902400T_{-4}T_{-8} - 12902400T_{-6}^2$$

$$J_{2,5} = -6451200T_{-4}T_{-6} + 6451200T_{-2}T_{-8}$$

$$J_{3,5} = -3225600T_{-4}^2 + 3225600T_{-8}T_0$$

$$J_{4,5} = -3225600T_{-4}T_{-2} + 1612800T_{-8}T_2 + 1612800T_{-6}T_0$$

$$J_{5,5} = 774144T_{-8}T_4 - 1290240T_{-2}^2 + 1806336T_{-6}T_2 - 1290240T_{-4}T_0$$

$$J_{6,5} = -322560T_{-4}T_2 - 1612800T_{-2}T_0 + 322560T_{-8}T_6 + 1612800T_{-6}T_4$$

$$J_{7,5} = 645120T_{-4}T_4 - 1935360T_{-2}T_2 + 1198080T_{-6}T_6 + 92160T_{-8}T_8$$

$$J_{8,5} = 645120T_{-6}T_8 - 2257920T_{-2}T_4 + 1612800T_{-4}T_6$$

$$J_{9,5} = -2580480T_{-2}T_6 + 2580480T_{-4}T_8$$

The λ component:

$$J_{1,6} = T_{-8}$$

$$J_{2,6} = T_{-6}$$

$$J_{3,6} = T_{-4}$$

$$J_{4,6} = T_{-2}$$

$$J_{5,6} = T_0$$

$$J_{6,6} = T_2$$

$$J_{7,6} = T_4$$

$$J_{8,6} = T_6$$

$$J_{9,6} = T_8$$

The φ_{-2}, φ_0, and φ_2 components:

$$\begin{aligned}
J_{1,7} &= -16T_{-6}, & J_{1,8} &= 16T_{-8}, & J_{1,9} &= 0 \\
J_{2,7} &= -14T_{-4}, & J_{2,8} &= 12T_{-6}, & J_{2,9} &= 2T_{-8} \\
J_{3,7} &= -12T_{-2}, & J_{3,8} &= 8T_{-4}, & J_{3,9} &= 4T_{-6} \\
J_{4,7} &= -10T_0, & J_{4,8} &= 4T_{-2}, & J_{4,9} &= 6T_{-4} \\
J_{5,7} &= -8T_2, & J_{5,8} &= 0, & J_{5,9} &= 8T_{-2} \\
J_{6,7} &= -6T_4, & J_{6,8} &= -4T_2, & J_{6,9} &= 10T_0 \\
J_{7,7} &= -4T_6, & J_{7,8} &= -8T_4, & J_{7,9} &= 12T_2 \\
J_{8,7} &= -2T_8, & J_{8,8} &= -12T_6, & J_{8,9} &= 14T_4 \\
J_{9,7} &= 0, & J_{9,8} &= -16T_8, & J_{9,9} &= 16T_6
\end{aligned}$$

Bibliography

[Ahl78] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR MR510197 (80c:30001) 57

[BCG⁺91] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991. MR MR1083148 (92h:58007) 1, 15, 44, 49

[BFT08] P. A. Burovskiy, E. V. Ferapontov, and S. P. Tsarev, *Second order quasilinear pdes and conformal structures in projective space*, arXiv:0802.2626v3 [nlin.SI] (2008). 108

[BGG03] Robert Bryant, Phillip Griffiths, and Daniel Grossman, *Exterior differential systems and Euler-Lagrange partial differential equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2003. MR MR1985469 (2004g:58001) 15

[Bli17] Hans Frederik Blichfeldt, *Finite collineation groups: with an introduction to the theory of groups of operators and substitution groups*, The University of Chicago Press, 1917. 61

[BLP03] V. M. Bukhshtaber, D. V. Leĭkin, and M. V. Pavlov, *Egorov hydrodynamic chains, the Chazy equation, and the group* $\mathrm{SL}(2,\mathbb{C})$, 2003. MR MR2083228 (2005h:37151) 13

[BN07] Marcin Bobienski and Pawel Nurowski, *Irreducible SO(3) geometry in dimension five*, arXiv:0507152 [math.DG] (2007). 2, 35, 109

[BO00] Irina Berchenko and Peter J. Olver, *Symmetries of polynomials*, J. Symbolic Comput. **29** (2000), no. 4-5, 485–514, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). MR MR1769652 (2001g:13007) 57, 59, 61, 63, 67, 71, 74

[Bry91] Robert L. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, 33–88. MR MR1141197 (93e:53030) 2, 3, 35, 107

[Bry01] ———, *Bochner-Kähler metrics*, J. Amer. Math. Soc. **14** (2001), no. 3, 623–715 (electronic). MR MR1824987 (2002i:53096) 15, 23

[Car04] Élie Cartan, *Sur la structure des groupes infinis de transformation*, Ann. Sci. École Norm. Sup. (3) **21** (1904), 153–206. MR MR1509040 15, 23

[Car41] E. Cartan, *La geometria de las equaciones diferenciales de tercer orden 131. reprinted in: E. cartan, oeuvres compltes, partie iii-2, editions du cnrs, 1984, pp. 15351566.*, Oeuvres Completes, Partie III-2, Editions du CNRS, 1984, pp. 15351566., Rev. Mater. Hispano-Am, 1941. 2

[CF03] Marius Crainic and Rui Loja Fernandes, *Integrability of Lie brackets*, Ann. of Math. (2) **157** (2003), no. 2, 575–620. MR MR1973056 (2004h:58027) 16, 23

[Che40] Shiing-shen Chern, *The geometry of the differential equation $y''' = F(x, y, y', y'')$*, Sci. Rep. Nat. Tsing Hua Univ. (A) **4** (1940), 97–111. MR MR0004538 (3,21c) 2

[Chu06] Kenneth C. Chu, *The moduli space of real binary octics*, Ph.D. thesis, The University of Utah, 2006. 57, 64

[Chu07] ———, *On the geometry of the moduli space of real binary octics*, arXiv:0708.0419v1 [math.AG] (2007). 57, 64

[CM09] Peter Crooks and Robert Milson, *On projective equivalence of univariate polynomial subspaces*, arXiv:0902.1106v2 [math.QA] (2009). 34, 51

[Con78] John B. Conway, *Functions of one complex variable*, second ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978. MR MR503901 (80c:30003) 57

[COP05] Jeongoo Cheh, Peter J. Olver, and Juha Pohjanpelto, *Maurer–cartan equations for lie symmetry pseudogroups of differential equations*, Journal of Mathematical Physics **46** (2005), no. 2, 023504. 108

[CS07] Andreas Cap and Katja Sagerschnig, *On nurowski's conformal structure associated to a generic rank two distribution in dimension five*, arXiv:0710.2208v1 [math.DG] (2007). 2

[Dou01] Boris Doubrov, *Contact trivialization of ordinary differential equations*, 73–84. MR MR1978764 (2004c:34111) 2

[DT06] Maciej Dunajski and Paul Tod, *Paraconformal geometry of nth order odes, and exotic holonomy in dimension four*, J.Geom.Phys. **56** (2006), 1790–1809. 2

[Ell96] Edwin Bailey Elliott, *Literatur-Berichte: An Introduction to the algebra of quantics*, Monatsh. Math. Phys. **7** (1896), no. 1, A31–A32, By Edwin Bailey Elliott, M. A., F. R. S., Wainflete Professor of pure mathematics, and Fellow of Magdalen College, Oxford. XIII+423 S. 8 °. At the Clarendon press, Oxford. 1895. Price 15 sh. MR MR1546431 56

[FC06] Rui Loja Fernandes and Marius Crainic, *Lectures on integrability of lie brackets*, arXiv::0507152 [math.DG] (2006). 23, 24, 25

[FHK07] E. V. Ferapontov, L. Hadjikos, and K. R. Khusnutdinova, *Integrable equations of the dispersionless hirota type and hypersurfaces in the lagrangian grassmannian*, arXiv:0705.1774v1 [math.DG] (2007). 2, 3, 4, 5, 12, 13, 14, 35, 45, 51, 81, 88, 108

[FK03] E. V. Ferapontov and K. R. Khusnutdinova, *Hydrodynamic reductions of multi-dimensional dispersionless PDEs: the test for integrability*, arXiv:nlin/0212015 (2003). 13

[FO07] E. V. Ferapontov and A. V. Odesskii, *Integrable lagrangians and modular forms*, arXiv:0707.3433v2 [nlin.SI] (2007). 13

[Gar89] Robert B. Gardner, *The method of equivalence and its applications*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 58, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989. MR MR1062197 (91j:58007) 15

[GN06] Michal Godlinski and Pawel Nurowski, *Third-order ODEs and four-dimensional split signature Einstein metrics*, J. Geom. Phys. **56** (2006), no. 3, 344–357. MR MR2171889 (2006k:53118) 2, 35

[GN07] Michal Godlinski and Pawel Nurowski, *GL(2,R) geometry of ODE's*, arXiv:0710.0297v1 [math.DG] (2007). 2, 35

[GN09] ———, *Geometry of third-order ODEs*, arXiv:0902.4129v1 [math.DG] (2009). 2, 35

[GS87] R. B. Gardner and W. F. Shadwick, *A simple characterization of the contact system on $J^k(E)$*, Rocky Mountain J. Math. **17** (1987), no. 1, 19–21. MR MR882881 (88g:58007) 6

[GT96] John Gibbons and Serguei P. Tsarev, *Reductions of the Benney equations*, Phys. Lett. A **211** (1996), no. 1, 19–24. MR MR1372470 (96j:35251) 13

[Har95] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR MR1416564 (97e:14001) 9, 10

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR MR1631825 (99g:14031) 56

[Hum72] James E. Humphreys, *Introduction to Lie algebras and representation theory*, xii+169, Graduate Texts in Mathematics, Vol. 9. MR MR0323842 (48 #2197) 31

[IL03] Thomas A. Ivey and J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI, 2003. MR MR2003610 (2004g:53002) 15, 44, 82

[KG89] Yuji Kodama and John Gibbons, *A method for solving the dispersionless KP hierarchy and its exact solutions. II*, Phys. Lett. A **135** (1989), no. 3, 167–170. MR MR985513 (90i:35248) 13

[Kod88] Yuji Kodama, *A method for solving the dispersionless KP equation and its exact solutions*, Phys. Lett. A **129** (1988), no. 4, 223–226. MR MR944250 (89f:35186) 13

[LR98] Ian G. Lisle and Gregory J. Reid, *Geometry and structure of Lie pseudogroups from infinitesimal defining systems*, J. Symbolic Comput. **26** (1998), no. 3, 355–379. MR MR1633880 (99h:58206) 108

[LR00] ______, *Cartan structure of infinite Lie pseudogroups*, Geometric approaches to differential equations (Canberra, 1995), Austral. Math. Soc. Lect. Ser., vol. 15, Cambridge Univ. Press, Cambridge, 2000, pp. 116–145. MR MR1761237 (2001e:58023) 108

[Mac95] Kirill C. H. Mackenzie, *Lie algebroids and Lie pseudoalgebras*, Bull. London Math. Soc. **27** (1995), no. 2, 97–147. MR MR1325261 (96i:58183) 23

[Mac05] ______, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005. MR MR2157566 (2006k:58035) 23, 27

[MY93] Keiji Matsumoto and Masaaki Yoshida, *Configuration space of* 8 *points on the projective line and a* 5*-dimensional Picard modular group*, Compositio Math. **86** (1993), no. 3, 265–280. MR MR1219628 (94c:14038) 57, 64

[Nur07] Pawel Nurowski, *Comment on GL(2,R) geometry of 4th order ODEs*, arXiv:0710.1658v1 [math.DG] (2007). 2, 35, 109

[Olv90] Peter J. Olver, *Classical invariant theory and the equivalence problem for particle Lagrangians. I. Binary forms*, Adv. Math. **80** (1990), no. 1, 39–77. MR MR1041883 (91c:58002) 56, 59, 61

[Pav03] Maxim V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), no. 9, 4134–4156. MR MR2003950 (2004h:37107) 13

[PWZ96] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, $A = B$, A K Peters Ltd., Wellesley, MA, 1996, With a foreword by Donald E. Knuth, With a separately available computer disk. MR MR1379802 (97j:05001) 110

[Ste74] P. Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. London Math. Soc. (3) **29** (1974), 699–713. MR MR0362395 (50 #14837) 26

[Ste80] ______, *Integrability of systems of vector fields*, J. London Math. Soc. (2) **21** (1980), no. 3, 544–556. MR MR577729 (81h:49026) 26

[Sus73a] Héctor J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188. MR MR0321133 (47 #9666) 26

[Sus73b] ______, *Orbits of families of vector fields and integrability of systems with singularities*, Bull. Amer. Math. Soc. **79** (1973), 197–199. MR MR0310922 (46 #10020) 26

[SW93] D. H. Sattinger and O. L. Weaver, *Lie groups and algebras with applications to physics, geometry, and mechanics*, Applied Mathematical Sciences, vol. 61, Springer-Verlag, New York, 1993, Corrected reprint of the 1986 original. MR MR1285814 (95i:22034) 24

[Tsa90] S. P. Tsarëv, *The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 5, 1048–1068. MR MR1086085 (92b:58109) 13

[Tsa93] ______, *Classical differential geometry and integrability of systems of hydrodynamic type*, Applications of analytic and geometric methods to nonlinear differential equations (Exeter, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 413, Kluwer Acad. Publ., Dordrecht, 1993, pp. 241–249. MR MR1261665 (94m:58129) 13

[Tsa00] ______, *Integrability of equations of hydrodynamic type from the end of the 19th to the end of the 20th century*, Integrability: the Seiberg-Witten and Whitham equations (Edinburgh, 1998), Gordon and Breach, Amsterdam, 2000, pp. 251–265. MR MR1816180 (2002c:37001) 13

[ZKR96] Daniel Zwillinger, Steven G. Krantz, and Kenneth H. Rosen (eds.), *CRC standard mathematical tables and formulae*, th ed., CRC Press, Boca Raton, FL, 1996. MR MR1392960 11

Biography

Abraham David Smith was born February 20, 1981 in Milwaukee, Wisconsin, USA. He graduated from University of Wisconsin—Madison in May, 2003 with a Bachelor of Science in Physics and Mathematics, with honors in the major in mathematics. Immediately thereafter, he entered the mathematics graduate program at Duke University, receiving a Master of Arts in December 2004 and a Doctor of Philosophy in Mathematics in May, 2009 (pending this dissertation). In July 2004, he married Kathryn Helen Condon.